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ON THE ASSOCIATED PRIMES OF THE GENERALIZED d-LOCAL COHOMOLOGY MODULES

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Abstract. The first part of the paper is concerned to relationship between the sets of associated primes of the generalized d-local cohomology modules and the ordinary generalized local cohomology modules. Assume that R is a commutative Noetherian local ring, M and N are finitely generated R-modules and d, t are two integers. We prove that $\operatorname{Ass} H^t_d(M,N) = \bigcup_{I\in\Phi} \operatorname{Ass} H^t_I(M,N)$ whenever $H^i_d(M,N) = 0$ for all $i < t$ and $\Phi = \{I : I \text{ is an ideal of } R \text{ with } \dim R/I \leq d\}$. In the second part of the paper, we give some information about the non-vanishing of the generalized d-local cohomology modules. To be more precise, we prove that $H^i_d(M,R) \neq 0$ if and only if $i = n - d$ whenever R is a Gorenstein ring of dimension n and $pd_R(M) < \infty$. This result leads to an example which shows that $\operatorname{Ass} H^{n-d}_d(M,R)$ is not necessarily a finite set.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity. For an ideal I of R and R-modules M and N, the ith generalized local cohomology module of

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M and N with respect to I is defined as

$$H^i_I(M,N) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/I^nM,N).$$

The reader can refer to [9, 18] for the basic properties of generalized local cohomology modules. An important problem in commutative algebra is determining when the set of associated primes of the ith local cohomology module $H^i_I(M)$ is finite. In [10], raised the following question: If M is a finitely generated R-module, then the set of associated primes of $H^i_I(M)$ is finite for all ideals I of R and all $i \geq 0$. In [11, 15, 17] the authors have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see [4, 5, 8, 13]. The purpose of this paper is to make a counterexample to above question in the context of general local cohomology modules. The theory of general local cohomology modules over commutative Noetherian ring introduced by M. H. Bijan-Zadeh in [3]. General local cohomology theory described as follows.

Let Φ be a non-empty set of ideals of R. We call Φ a system of ideals of R if, whenever $I, I' \in \Phi$, then there exists $J \in \Phi$ such that $J \subseteq II'$. Such a system of ideals gives rise to an additive, left exact functor

$$\Gamma_\Phi(M) = \{x \in M : Ix = 0 \ \text{ for some ideal } \ I \in \Phi\}$$

from the category of R-modules and R-homomorphisms to itself. $\Gamma_\Phi(-)$ is called the Φ-torsion functor. For each $i \geq 0$, the ith right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. For an ideal I of R, if $\Phi = \{I^n : n \in \mathbb{N}\}$, then $H^i_\Phi(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$. Let $d \geq 0$ be an integer. We denote $\Gamma_\Phi(-)$ and $H^i_\Phi(-)$ by $\Gamma_d(-)$ and $H^i_d(-)$ respectively, for the system of ideals $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$. The functor $\Gamma_d(-)$ was originally define in [1] and the modules $H^i_d(M)$ were called d-local cohomology modules associated to M. For $i \geq 0$, we define $H^i_d(-,-) : \mathfrak{C}(R) \times \mathfrak{C}(R) \to \mathfrak{C}(R)$ with

$$H^i_d(M,N) = \varinjlim_{I\in\Phi} \mathrm{Ext}^i_R(M/IM,N),$$

and call it the ith generalized d-local cohomology module of M and N. Then $H^i_d(-,-)$ is an additive, R- linear functor which is contravariant in the first variable and covariant in the second variable. After some preliminary results in section 2, for a finitely generated module M with $pd_R(M) < \infty$ and an integer t we prove that

$$\mathrm{Ass}(H^t_d(M,N)) = \bigcup_{I\in\Phi} \mathrm{Ass}(\mathrm{Ext}^t_R(M/IM,N)) = \bigcup_{I\in\Phi} \mathrm{Ass}(H^t_I(M,N)),$$

where $\Phi = \{I : I \text{ is an ideal of } R \text{ with} \dim R/I \leq d\}$ and $H^i_d(M,N) = 0$ for all $i < t$. In section 3, we shall provide some results concerning vanishing and non-vanishing of generalized d-local cohomology modules: we shall prove that, over a local ring R, if the non-zero finitely generated R-module M has Krull dimension n and $pd_R(M) < \infty$, then there exists an integer i with $0 \leq i \leq d$ such that $H_d^{n-i}(M, R) \neq 0$. We shall also prove that, when R is local Gorenstein of dimension n, then $H_d^i(M, R) \neq 0$ if and only if $i = n - d$. Furthermore, we shall prove that $H_d^{n-d}(M, R)$ is a non-Artinian module for which $\mathrm{Ass}(H_d^{n-d}(M, R)) = \{\mathfrak{p} \in \mathrm{Supp}(M) : \dim R/\mathfrak{p} = d\}$.

2. The Associated Primes

It is our intention in this section to present the relationship between the sets of associated primes of the generalized d-local cohomology modules and the ordinary generalized local cohomology modules. Its potential for use in arguments that employ the Grothendieck spectral sequence. So in this section, $(R, \mathfrak{m})$ is a local ring, I is an ideal of R, d is an integer and $\Phi = \{I : I \text{ is an ideal of R with} \dim R/I \leq d\}$.

We say that M is a d-torsion module if $\Gamma_d(M) = M$, and it is d-torsion free if $\Gamma_d(M) = 0$.

Lemma 2.1. *Let M and N be finitely generated R-modules and t be an integer such that $H_d^i(M, N) = 0$ for all $i < t$. Then the following statements are true:*
(i) $H_I^i(M, N) \subseteq H_d^i(M, N)$ for all $I \in \Phi$ and $i \leq t$.
(ii) $H_c^i(M, N) \subseteq H_d^i(M, N)$ for all $c \leq d$ and $i \leq t$.

Proof. (i) It is clear that $\mathrm{Hom}_R(R/I, \Gamma_d(M, N)) \cong \mathrm{Hom}_R(M/IM, N)$ for $I \in \Phi$ so by [16, Theorem 11.38], the Grothendieck spectral sequence $E_2^{p,q} := \mathrm{Ext}_R^p(R/I, H_d^q(M, N))$ converges to $E^{p+q} := \mathrm{Ext}_R^{p+q}(M/IM, N)$. Because $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ so $E_\infty^{p,q} = 0$, for all $q < t$. On the other hand, there is a finite filtration

$$0 = F^{q+1}E^q \subseteq F^qE^q \subseteq \cdots \subseteq F^1E^q \subseteq F^0E^q = E^q$$

of E^q such that $E_\infty^{p,q-p} \cong F^pE^q/F^{p+1}E^q$, for all $p = 0, 1, \cdots, q$. Thus $E_\infty^{0,q} \cong E^q$, for all $q \leq t$. Also, for all $q \leq t$, the sequence $0 \longrightarrow E_2^{0,q} \longrightarrow E_2^{2,q-1}$ and $E_2^{2,q-1} = 0$ imply that $E_\infty^{0,q} \cong E_2^{0,q}$, for all $q \leq t$. Thus $E^q \cong E_2^{0,q}$. Hence, $\mathrm{Ext}_R^q(M/IM, N) \cong \mathrm{Hom}_R(R/I, H_d^q(M, N))$. Therefore,

$$\Gamma_I(H_d^q(M, N)) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Hom}(R/I^n, H_d^q(M, N)) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^q(M/I^nM, N) \cong H_I^q(M, N).$$

The proof is therefore complete.

(ii) The proof is similar to that of (i). □

Lemma 2.2. *Let M and N be finitely generated R-modules and t be an integer such that $H_d^i(M, N) = 0$, for all $i < t$. Then the following statements are true:*
(i) $\mathrm{Ass}(H_I^i(M, N)) = \mathrm{Ass}(H_d^i(M, N)) \cap V(I)$, for all $I \in \Phi$ and $i \leq t$.
(ii) $\mathrm{Ass}(H_J^i(M, N)) \subseteq \mathrm{Ass}(H_I^i(M, N))$ for all $I, J \in \Phi$ with $I \subseteq J$ and $i \leq t$.

Proof. (i) Let $I \in \Phi$ and $i \leq t$. Then $\mathrm{Hom}_R(R/I, H^i_d(M,N)) \cong \mathrm{Hom}_R(R/I, H^i_I(M,N))$ and $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_d(M,N))) = \mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_I(M,N))) = \mathrm{Ass}(H^t_I(M,N))$, by the proof of Lemma 2.1 (i) and [12, Proposition 1.1]. Thus $\mathrm{Ass}(H^t_I(M,N)) = \mathrm{Ass}(H^t_d(M,N)) \cap V(I)$.

(ii) It is obvious by (i). □

Lemma 2.3. *If M is a d-torsion module, then* $\mathrm{Supp}(M) \subseteq \Phi$.

Proof. Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Then there exists $x \in M$ such that $\mathrm{Ann}(x) \subseteq \mathfrak{p}$. By assumption there is an ideal I of R such that $I \in \Phi$ and $Ix = 0$. Thus $I \subseteq \mathrm{Ann}(x) \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \Phi$. □

Theorem 2.4. *Let M and N be finitely generated R-modules and t be an integer such that $H^i_d(M,N) = 0$ for all $i < t$. Then*

$$\mathrm{Ass}(H^t_d(M,N)) = \bigcup_{I \in \Phi} \mathrm{Ass}(\mathrm{Ext}^t_R(M/IM, N)) = \bigcup_{I \in \Phi} \mathrm{Ass}(H^t_I(M,N)).$$

Proof. First of all, we show that

$$\mathrm{Ass}(H^t_d(M,N)) = \bigcup_{I \in \Phi} \mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_d(M,N))).$$

It is clear that $\mathrm{Hom}_R(R/I, H^t_d(M,N)) \cong 0 :_{H^t_d(M,N)} I$. Therefore, $\mathrm{Ass}(H^t_d(M,N)) \supseteq \bigcup_{I \in \Phi} \mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_d(M,N)))$. Let $\mathfrak{p} \in \mathrm{Ass}(H^t_d(M,N))$. Then there exists $0 \neq m \in H^t_d(M,N)$ such that $\mathfrak{p} = \mathrm{Ann}(m)$ now from Lemma 2.3 it follows that $\mathfrak{p} \in \Phi$. Thus $m \in 0 :_{H^t_d(M,N)} \mathfrak{p} \cong \mathrm{Hom}_R(R/\mathfrak{p}, H^t_d(M,N))$ and therefore $\mathrm{Ass}(H^t_d(M,N)) \subseteq \bigcup_{I \in \Phi} \mathrm{Ass}(\mathrm{Hom}_R(R/I, H^t_d(M,N)))$. On the other hand, by the proof of Lemma 2.1, $\mathrm{Hom}_R(R/I, H^t_d(M,N)) \cong \mathrm{Ext}^t_R(M/IM, N)$. Hence, the result follows. □

Theorem 2.5. *Let M and N be finitely generated R-modules and t be an integer such that $H^i_d(M,N) = 0$ for all $i < t$. Then $\mathrm{Ass}(H^t_d(M,N)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}(M) : \dim R/p = d\}$ if and only if $H^t_c(M,N) = 0$ for all integer c with $c < d$.*

Proof. Assume that $\mathrm{Ass}(H^t_d(M,N)) \nsubseteq \{\mathfrak{p} \in \mathrm{Supp}(M) : \dim R/\mathfrak{p} = d\}$. Thus there exists a prime ideal $\mathfrak{p}$ belongs to $\mathrm{Ass}(H^t_d(M,N))$ such that $\dim R/\mathfrak{p} = c < d$. So the exact sequence $0 \to \Gamma_c(R/\mathfrak{p}) \to \Gamma_c(H^t_d(M,N))$ and the fact that $\Gamma_c(R/\mathfrak{p}) = R/\mathfrak{p}$, $\Gamma_c(H^t_d(M,N)) \cong H^t_c(M,N)$ show that $\mathfrak{p} \in \mathrm{Ass}(H^t_c(M,N))$. Hence, $H^t_c(M,N) \neq 0$. The converse is true by Lemmas 2.3 and 2.1. □

3. The vanishing theorems

In this section, we shall provide some results concerning the vanishing and non-vanishing of generalized d-local cohomology modules.Throughout R is a local ring with maximal ideal $\mathfrak{m}$ and d is a non negative integer.

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring of dimension n and let M, N be finitely generated R-modules such that $\mathrm{Ass}_R(M) \cap \mathrm{Supp}(N) \neq \emptyset$ and $pd_R(M) < \infty$. Then there is at lest one j with $0 \leq j \leq d$ for which $H_d^{n-j}(M,N) \neq 0$.*

Proof. It is easy to see that $\Gamma_{\mathfrak{m}}(\Gamma_d(M,N)) = \Gamma_{\mathfrak{m}}(M,N)$ so there is a Grothendieck spectral sequence

$$E_2^{i,j} = H_{\mathfrak{m}}^i(H_d^j(M,N)) \Rightarrow E^{i+j} = H_{\mathfrak{m}}^{i+j}(M,N),$$

see [16, Theorem 11.38]. By Lemma 2.3, $\mathrm{Supp}(H_d^j(M,N)) \subseteq \Phi$. Thus $\dim H_d^j(M,N) \leq d$ and so $E_2^{i,j} = 0$ for all $i > d$, by [6, Theorem 6.1.2]. On the other hand, there is a finite filtration $0 = F^{n+1}E^n \subseteq F^nE^n \subseteq \cdots \subseteq F^1E^n \subseteq F^0E^n = E^n$ with $E_\infty^{i,n-i} \cong F^iE^n/F^{i+1}E^n$. Then $F^{d+1}E^n = \cdots = F^nE^n = 0$. If $E_\infty^{j,n-j} = 0$, for all j with $0 \leq j \leq d$, then $F^dE^n = \cdots = F^0E^n = E^n = 0$ contrary to [7, Lemma 2.4]. So suppose that $E_\infty^{j,n-j} \neq 0$ for some j with $0 \leq j \leq d$. Thus $E_2^{j,n-j} \neq 0$ and then $H_d^{n-j}(M,N) \neq 0$. □

Theorem 3.2. *Let R be a Gorenstein local ring of dimension n and M be a R- module with $pd_R(M) < \infty$. Then the following statements are true:*

(i) $H_d^i(M,R) = 0$ for all $0 \leq i < n-d$ and $H_d^i(M,R) \cong \mathrm{Ext}_R^i(M,R)$ for all $n-d \leq i \leq n$.

(ii) If $0 \leq d \leq \dim M$, then $\mathrm{Ass}(H_d^{n-d}(M,R)) = \{\mathfrak{p} \in \mathrm{Supp}(M) : \dim R/\mathfrak{p} = d\}$ and $H_d^{n-d}(M,R) \neq 0$.

(iii) If $0 < d \leq \dim M$, then $H_d^{n-d}(M,R)$ is a Noetherian module which is not Artinian.

Proof. (i) Let $0 \to R \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ be a minimal injective resolution of R. Then by [14, Theorems 18.1 and 18.8] we have $E^i \cong \bigoplus_{ht(\mathfrak{p})=i} E(R/\mathfrak{p}) = \bigoplus_{\dim R/\mathfrak{p}=n-i} E(R/\mathfrak{p})$. If $i < n-d$, then $\dim R/\mathfrak{p} = n-i > d$ and so $\mathfrak{p} \notin \Phi$. Thus $\Gamma_d(E^i) = 0$ and so $\mathrm{Hom}(M, \Gamma_d(E^i)) = 0$. It follows that $H_d^i(M,R) = 0$, for all $0 \leq i < n-d$. If $i \geq n-d$, then $\dim R/\mathfrak{p} = n-i \leq n-(n-d) = d$, and so $\mathfrak{p} \in \Phi$. Thus $\Gamma_d(E^i) = E^i$ and $\mathrm{Hom}(M, \Gamma_d(E^i)) = \mathrm{Hom}(M, E^i)$. It follows that $H_d^i(M,R) \cong \mathrm{Ext}_R^i(M,R)$ for all $n-d \leq i \leq n$.

(ii) It is immediate that

$$\begin{aligned} \mathrm{Ass}(H_d^{n-d}(M,R)) &\subseteq \mathrm{Ass}(\mathrm{Hom}_R(M, \bigoplus_{\dim R/\mathfrak{p}=d} E(R/\mathfrak{p}))) \\ &= \{\mathfrak{p} \in \mathrm{Supp}(M) : \dim R/\mathfrak{p} = d\}. \end{aligned}$$

If $\mathfrak{p} \in \operatorname{Supp}(M)$ and $\dim R/\mathfrak{p} = d$, then by [19, Lemma 2.1(5)], $(H_d^{n-d}(M,R))_\mathfrak{p} \cong H_0^{n-d}(M_\mathfrak{p}, R_\mathfrak{p}) \cong H_{\mathfrak{p}R_\mathfrak{p}}^{n-d}(M_\mathfrak{p}, R_\mathfrak{p})$ since every Gorenstein local ring is catenary and biequidimensional, see [14, Theorem 17.3]. Moreover, $H_{\mathfrak{p}R_\mathfrak{p}}^{n-d}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$ by [7, Lemma 2.4]. Hence, $\mathfrak{p}$ is a minimal element of $\operatorname{Supp}(H_d^{n-d}(M,R))$ and therefore $\mathfrak{p} \in \operatorname{Ass}(H_d^{n-d}(M,R))$.

(iii) By the proof of (ii), $\operatorname{Ass}(H_d^{n-d}(M,R)) \nsubseteq \operatorname{Max}(R)$ so that $H_d^{n-d}(M,R)$ is not Artinian.

□

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a local ring and M, N two finitely generated R- modules with $pd(M) < \infty$. If t is a non-negative integer such that $pd(M) < t$, then the following statements are equivalent:*

(i) $H_d^i(M,N)$ is finitely generated, for all $i \geq t$;

(ii) $H_d^i(M,N) = 0$, for all $i \geq t$.

Proof. (i)⇒ (ii). We use induction on $n = \dim N$. When $n = 0$, we have $H_d^i(M,N) = 0$, for all $i \geq t$, see [19, Theorem 4.1]. Assume inductively, that $n > 0$ and that the result has been proved for finitely generated R-modules of dimension $n-1$. The exact sequence

$$0 \to \Gamma_d(N) \to N \to N/\Gamma_d(N) \to 0$$

yields the long exact sequence

$$\cdots \to H_d^i(M, \Gamma_d(N)) \to H_d^i(M,N) \to H_d^i(M, N/\Gamma_d(N)) \to \cdots .$$

By [19, Lemma 2.1(3)], $H_d^i(M,\Gamma_d(N)) \cong \operatorname{Ext}^i(M,\Gamma_d(N))$, for all $i \geq 0$. Thus we have $H_d^i(M,\Gamma_d(N)) = 0$ and $H_d^i(M, N/\Gamma_d(N)) \cong H_d^i(M,N)$, for all $i > pd(M)$. Hence, we may assume, by replacing N with $N/\Gamma_d(N)$, that N is d-torsion free module. So there exists $x \in \mathfrak{m}$ which is a non-zero divisor on N. The exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

induces the long exact sequence

$$\cdots \to H_d^i(M,N) \xrightarrow{x} H_d^i(M,N) \to H_d^i(M,N/xN) \to H_d^{i+1}(M,N) \to \cdots .$$

which implies that $H_d^i(M,N/xN)$ is finitely generated, for all $i \geq t$. Since N/xN is a finitely generated R-module with $\dim N/xN = n-1$ thus by the inductive hypothesis $H_d^i(M,N/xN) = 0$, for all $i \geq t$. Therefore, $H_d^i(M,N) \cong xH_d^i(M,N)$, for all $i \geq t$. Hence, $H_d^i(M,N) = 0$, for all $i \geq t$, by Nakayma's Lemma.

(ii)⇒ (i). It is clear. □

4. The Artinianness theorems

Theorem 4.1. *Let M be a finitely generated R-module and N be an Artinian R- module, then $H_d^i(M, N)$ is Artinian, for all $i \geq 0$.*

Proof. We are going to argue by induction on i. In the case where $i = 0$, we see from $\Gamma_d(M, N) = \mathrm{Hom}(M, \Gamma_d(N))$ that $H_d^0(M, N)$ is Artinian. Now suppose, inductively, that $i > 0$ and that the result has been proved for all integers less than i. Let $E(N)$ be the injective envelope of N. We have the exact sequence

$$0 \to N \to E(N) \to E(N)/N \to 0$$

which induces a long exact sequence

$$\cdots \to H_d^{i-1}(M, E(N)/N) \to H_d^i(M, N) \to H_d^i(M, E(N)) \to \cdots .$$

Since $H_d^i(M, E(N)) = 0$, for all $i > 0$, thus $H_d^{i-1}(M, E(N)/N) \cong H_d^i(M, N)$, for all $i > 1$. By hypothesis N is Artinian so $E(N)$ is Artinian thus by induction hypothesis $H_d^{i-1}(M, E(N)/N)$ is Artinian. Hence, $H_d^i(M, N)$ is Artinian. This completes the inductive step. □

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a local ring and M, N be two finitely generated R- modules with $r = pd(M)$ and $n = \dim(N)$. Then*

$$H_d^{r+n}(M, N) \cong \mathrm{Ext}_R^r(M, H_d^n(N)).$$

In particular, $H_d^{r+n}(M, N)$ is an Artinian R-module.

Proof. By [16, Theorem 11.38], there is a Grothendieck spectral sequence,

$$E_2^{p,q} := \mathrm{Ext}_R^p(M, H_d^q(N)) \Rightarrow E^{p+q} := H_d^{p+q}(M, N).$$

We have $H_d^q(N) = 0$, for all $q > n$, see [3, Lemma 2.1] and [6, Theorem 6.1.2]. Then $E_2^{p,q} = 0$, for all $p > r$ or $q > n$. We have $E_k^{r-k,n+k-1} = E_k^{r+k,n+1-k} = 0$, for all $k \geq 2$, and homomorphisms of the spectral

$$E_k^{r-k,n+k-1} \to E_k^{r,n} \to E_k^{r+k,n+1-k}$$

so $E_2^{r,n} = E_3^{r,n} = \cdots = E_\infty^{r,n}$. It is enough to prove that $E_\infty^{r,n} \cong H_d^{r+n}(M, N)$. There is a filtration

$$0 = F^{r+n+1}H^{r+n} \subseteq \cdots \subseteq F^1 H^{r+n} \subseteq F^0 H^{r+n} = H_d^{r+n}(M, N)$$

and

$$E_\infty^{i,r+n-i} \cong F^i H^{r+n}/F^{i+1}H^{r+n}$$

for all $0 \leq i \leq r+n$. Thus $E_2^{i,r+n-i} = \mathrm{Ext}_R^i(M, H_d^{r+n-i}(N)) = 0$ for all $i \neq r$. Hence

$$F^{r+1}H^{r+n} = F^{r+2}H^{r+n} = \cdots = F^{r+n+1}H^{r+n} = 0$$

and

$$F^{r}H^{r+n} = F^{r-1}H^{r+n} = \cdots = F^{n}H^{r+n} = H_d^{r+n}(M, N).$$

This gives

$$E_\infty^{r,n} \cong F^rH^{r+n}/F^{r+1}H^{r+n} \cong H_d^{r+n}(M, N).$$

So $\mathrm{Ext}_R^r(M, H_d^n(N)) \cong H_d^{r+n}(M, N)$. By [2, Theorem 3.1], $H_d^n(N)$ is an Artinian R-module, therefore $H_d^{r+n}(M, N)$ is also Artinian. □

Theorem 4.3. *Let M be a finitely generated R-module and N be an R-module. Let t be a non-negative integer such that $H_d^i(N)$ is Artinian for all $i < t$. Then the following statements are true:*

(i) $H_d^i(M, N)$ is Artinian, for all $i < t$.

(ii) $\mathrm{Ext}_R^i(R/\mathfrak{a}, N)$ is Artinian, for all $i < t$ and for all $\mathfrak{a} \in \Phi$.

Proof. (i) We use induction on t. When $t = 1$, $\mathrm{Hom}(M, \Gamma_d(N))$ is Artinian so that $\Gamma_d(M, N)$ is Artinian since $\Gamma_d(M, N) = \mathrm{Hom}(M, \Gamma_d(N))$, see [19, Lemma 2.1(1)]. Now suppose that $t > 1$ and that the result has been proved for each integer less than t. Let $E(N)$ be the injective envelope of N. Then we have the exact sequence

$$0 \to N \to E(N) \to E(N)/N \to 0.$$

Applying the functors $\Gamma_d(-)$ and $\Gamma_d(M, -)$, we get isomorphisms $H_d^i(E(N)/N) \cong H_d^{i+1}(N)$ and $H_d^i(M, E(N)/N) \cong H_d^{i+1}(M, N)$ for all $i > 0$. From the hypothesis, $H_d^i(N)$ is Artinian for all $i < t$, thus $H_d^i(E(N)/N)$ is Artinian for all $i < t-1$. By the inductive hypothesis on $E(N)/N$, $H_d^i(M, E(N)/N)$ is Artinian, for all $i < t-1$. Then by second isomorphism $H_d^i(M, N)$ is Artinian, for all $i < t$.

(ii) We use induction on t. When $t = 1$, we have the exact sequence

$$0 \to \Gamma_d(N) \to N \to N/\Gamma_d(N) \to 0$$

thus there is a long exact sequence

$$0 \to \mathrm{Hom}(R/\mathfrak{a}, \Gamma_d(N)) \to \mathrm{Hom}(R/\mathfrak{a}, N) \to \mathrm{Hom}(R/\mathfrak{a}, N/\Gamma_d(N)) \to \cdots.$$

As $N/\Gamma_d(N)$ is Φ-torsion-free, $\mathrm{Hom}(R/\mathfrak{a}, N/\Gamma_d(N)) = 0$ and

$$\mathrm{Hom}(R/\mathfrak{a}, N) \cong \mathrm{Hom}(R/\mathfrak{a}, \Gamma_d(N)).$$

Since $\Gamma_d(N)$ is Artinian R-module, then $\mathrm{Hom}(R/\mathfrak{a}, \Gamma_d(N))$ is Artinian R-module. The proof for $t > 1$ is similar to that of (i). □

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