A CHARACTERIZATION OF SOME SIMPLE UNITARY GROUPS VIA ORDER AND DEGREE PATTERN OF SOLVABLE GRAPH

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ABSTRACT. The solvable graph associated with a finite group $G$, denoted by $\Gamma_s(G)$, is a simple graph whose vertices are the prime divisors of $|G|$ and two distinct primes $p$ and $q$ are joined by an edge if and only if there exists a solvable subgroup of $G$ whose order is divisible by $pq$. In this paper, we give a characterization for projective special unitary groups $U_3(q)$ with some certain conditions by the solvable graph.

1. INTRODUCTION

All groups appearing here are assumed to be finite. For a finite group $G$, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. There are a lot of ways for studying a finite group. One of the most interesting approaches is to consider some properties of the graphs associated with it. In fact, one of the graphs associated with a finite group $G$ is the solvable graph which was introduced by Abe and Iiyori in [2]. This graph is a simple and undirected graph that constructs as follows. The vertex set is $\pi(G)$, and two distinct primes $p$ and $q$ are adjacent

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(we write \( p \approx q \)) if and only if \( G \) has a solvable subgroup whose order is divisible by \( pq \). We denote this graph by \( \Gamma_s(G) \).

The set of orders of all elements in a finite group \( G \) is denoted by \( \text{Spec}(G) \) and called the spectrum of \( G \). This set is closed and partially ordered by the divisibility relation; therefore, it is determined uniquely from the subset \( \mu(G) \) of all maximal elements of \( \text{Spec}(G) \) with respect to divisibility. According to the definition of solvable graph, for the element \( a \) in \( \mu(G) \), any two primes \( p \) and \( q \) such that \( pq | a \), are joined by an edge.

Let \( p_1 < p_2 < \cdots < p_k \) be all primes in \( \pi(G) \). We define
\[
D_s(G) = (d_s(p_1), d_s(p_2), \ldots, d_s(p_k)),
\]
as the degree pattern of the solvable graph of \( G \), where \( d_s(p) \) signifies the degree of the vertex \( p \) in \( \Gamma_s(G) \).

We will give more attention to a prime in \( \Gamma_s(G) \) which is joined to any other primes in the graph that is called a regular prime. In other words, a prime \( p \) is a regular prime if \( d_s(p) = k - 1 \) where \( |\pi(G)| = k \). We denote the set of regular primes by \( \text{Reg}(G) \).

Given a finite group \( G \), denote by \( h_{OD_s}(G) \) the number of isomorphism classes of finite groups \( H \) such that \( |H| = |G| \) and \( D_s(H) = D_s(G) \). In terms of the function \( h_{OD_s}(\cdot) \), we have the following definition.

**Definition 1.1.** A finite group \( G \) is said to be \( n \)-fold OD\(_s\)-characterizable if \( h_{OD_s}(G) = n \). The group \( G \) is OD\(_s\)-characterizable if \( h_{OD_s}(G) = 1 \). Moreover, we will simply say that \( H \) is OD\(_s\)-characterizable if \( H \cong G \) for every finite group \( G \) such that \( |G| = |H| \) and \( D_s(G) = D_s(H) \).

In this paper, we are interested in characterizing groups by order and degree pattern of solvable graph. In \([3, 4]\), some simple groups were considered and shown that the following groups are OD\(_s\)-characterizable.

- All sporadic simple groups;
- Projective special linear groups \( L_2(q) \) with the following conditions:
  (a) \( p = 2 \), \( |\pi(q + 1)| = 1 \) or \( |\pi(q - 1)| = 1 \),
  (b) \( q \equiv 1 \pmod{4} \), \( |\pi(q + 1)| = 2 \) or \( |\pi(q - 1)| \leq 2 \),
  (c) \( q \equiv -1 \pmod{4} \).
- Projective special linear groups \( L_3(q) \) with the following conditions:
  (a) \( q \) is odd and \( 9 \nmid q - 1 \);
  (b) \( q \) is even and \( 3 \nmid q - 1 \);
  (c) \( 9 | q - 1 \) and \( |\pi(\frac{q^2 + q + 1}{3})| = 1 \);
  (d) \( q \) is even, \( 3 | q + 1 \) and \( |\pi(q^2 + q + 1)| = 1 \).
- A finite group \( H \) such that \( |\pi(H)| \geq 3 \), \( H \notin \{B_n(q), C_n(q)\} \) (\( n \geq 3 \) and \( q \) is odd), and \( \text{Reg}(H) = \emptyset \).
We will show that the projective special unitary groups $U_3(q)$ defined over a field of characteristic 3 with certain conditions, are OD$_s$-characterizable. In fact, we prove the following theorem.

**Theorem.** The simple groups $U_3(q)$ where $q = 3^f$ ($f \geq 2$), such that $|\pi(q^2 - q + 1)| = 1$ are OD$_s$-characterizable.

2. Preliminaries

We begin this section with some obtained results on solvable graphs of finite groups which will be used for further studies.

**Lemma 2.1.** ([3, Lemma 3]) The solvable graph of a finite group is always a connected graph.

**Lemma 2.2.** ([2]) Let $G$ be a non-abelian simple group. Then $\Gamma_s(G)$ is an incomplete graph.

**Lemma 2.3.** ([1, Lemma 3]) Let $G$ be a finite group such that $\text{Reg}(G) = \emptyset$. Then $G$ is a non-abelian simple group.

**Lemma 2.4.** ([2, Lemma 2]) Let $G$ be a group, $H$ a subgroup of $G$ and $N$ a normal subgroup of $G$. Then

1. If $p$ and $q$ are joined in $\Gamma_s(H)$ for $p, q \in \pi(H)$, then $p$ and $q$ are joined in $\Gamma_s(G)$, that is, $\Gamma_s(H)$ is a subgraph of $\Gamma_s(G)$.
2. If $p$ and $q$ are joined in $\Gamma_s(G/N)$ for $p, q \in \pi(G/N)$, then $p$ and $q$ are joined in $\Gamma_s(G)$, that is, $\Gamma_s(G/N)$ is a subgraph of $\Gamma_s(G)$.
3. For $p \in \pi(N)$ and $q \in \pi(G) \setminus \pi(N)$, $p$ and $q$ are joined in $\Gamma_s(G)$.

**Lemma 2.5.** ([3, Corollary 1]) Let $N$ be a normal Hall subgroup of a finite group $G$. Then $\Gamma_s(G)$ is complete if and only if $\Gamma_s(N)$ and $\Gamma_s(G/N)$ are complete, too.

**Lemma 2.6.** Let $K$ be a finite group and $G$ a subgroup of $K$ which is a simple group with $|K : G| = 2$. Then we have:

$$\Gamma_s(K) - \{2\} = \Gamma_s(G) - \{2\}.$$  

In particular, if $r \in \pi(G) - \{2\}$, then $d_{s_G}(r) \leq d_{s_K}(r) \leq d_{s_G}(r) + 1$, and moreover; if 2 is a regular prime in $\Gamma_s(G)$, then $d_{s_K}(r) = d_{s_G}(r)$.

**Proof.** We first claim that every subgroup of $K$ of odd order is a subgroup of $G$. Suppose that the claim is false and there exists a subgroup $H \leq K$ of odd order such that $H \not\leq G$. Thus there is an element $x \in K \setminus G$ such that $o(x) = m$ where $m$ is an odd number. Then we have $x^{-1} = x^{m-1} \in G$ since $|K : G| = 2$ and $m - 1$ is even. It follows that $x \in G$, which is a contradiction.
Note that $\pi(K) = \pi(G)$. In what follows, we will show that, if $p$ and $q$ are two odd primes such that $p \approx q$ in $\Gamma_s(K)$, then $p \approx q$ in $\Gamma_s(G)$. Assume that $p \approx q$ in $\Gamma_s(K)$. Hence, there is a solvable subgroup $L \leq K$ such that $pq | |L|$. We consider Hall $\{p, q\}$-subgroup $H$ of $L$. Now from the previous paragraph of the proof, $H$ is a subgroup of $G$ and so $p \approx q$ in $\Gamma_s(G)$. □

The following lemma is a fundamental result which is gained in \[2\].

**Lemma 2.7.** ([2, Lemma 1]) The solvable graph of a solvable group is complete.

We are not sure if the converse of Lemma 2.7 holds. It can be shown that it is true in some special cases. So we can state the following conjecture.

**Conjecture.** Let $G$ be a finite group. Then $\Gamma_s(G)$ is complete if and only if $G$ is a solvable group.

In the following lemma, we prove that if the group $G$ has a normal Hall subgroup, the conjecture above is true.

**Lemma 2.8.** Let $G$ be a finite group possessing a normal Hall subgroup. Then $\Gamma_s(G)$ is complete if and only if $G$ is a solvable group.

**Proof.** It is enough to consider the necessity. Suppose that $G$ is a counterexample of minimal order to this statement which means that $\Gamma_s(G)$ is complete while $G$ is not solvable. Assume that $N$ is the normal Hall subgroup of $G$. Now we obtain from Lemma 2.7 that $\Gamma_s(N)$ and $\Gamma_s(G/N)$ are complete. Thus we have by the hypothesis that $N$ and $G/N$ are solvable. Consequently, $G$ is solvable which contradicts our assumption. □

**Lemma 2.9.** ([2, Theorem 3]) Let $G$ be a finite group and $\{p, q\} \subseteq \pi(G)$. Then $p$ and $q$ are not joined in $\Gamma_s(G)$ if and only if there exists a series of normal subgroups of $G$, say

$$1 \leq M < N \leq G,$$

such that $M$ and $G/N$ are $\{p, q\}$-groups and $N/M$ is a non-abelian simple group such that $p$ and $q$ are not joined in $\Gamma_s(N/M)$.

Using the notation taken from [1] and [2], such a series as in Lemma 2.9 is called a GKS-series of $G$ and we will say $p$ and $q$ are expressed to be disjoint by this GKS-series.

**Lemma 2.10.** ([1, Lemma 4]) Let $G$ be a finite group with $|\pi(G)| = k$. If the number of connected components of

$$\tilde{\Gamma}(G) = (\Gamma_s(G) - \text{Reg}(G))^c$$

equals to $n$, then at most $n$ GKS-series of $G$ is necessary to express any pair of vertices of $\Gamma_s(G)$ to be disjoint.
The following lemma is a result of Lemma 2.10.

**Lemma 2.11.** [3, Lemma 6] Let $G$ be a finite group with $|\pi(G)| = k \geq 4$. Moreover, let $\text{Reg}(G) \neq \emptyset$ and

$$\Gamma(G) := (\Gamma_s(G) - \text{Reg}(G))^c.$$ 

If there is a prime $p \in \pi(G)$ such that $d_s(p) = 1$ or $2$, then any disjointed pair of vertices of $\Gamma_s(G)$ can be expressed by only one GKS-series.

We now present a lemma which will be used in Section 3.

**Lemma 2.12.** [10, Lemma 2] Let $q = p^f$, where $p$ is a prime and $f$ a natural number. If

$$|\pi((q - 1)/(2, q - 1))| \leq 2 \geq |\pi((q + 1)/(2, q - 1))|,$$

then $q = 4, 9, 16, 81$ or $q = p^f$, $f = 1$ or an odd prime.

In the sequel, we are going to construct the solvable graph of projective special unitary groups $U_3(q)$ where $q = 3^f$ and $f$ is a natural number. We draw this graph in a compact form. The compact form of a graph is a graph whose vertices are displayed with disjoint subsets of the vertex set in the graph. Actually, a vertex labeled $U$ represents the complete subgraph of the graph on $U$. An edge connecting $U$ and $W$ represents the set of edges in the graph which connect each vertex in $U$ with each vertex in $W$.

- The set of maximal elements in the spectrum of $U_3(q)$ is as follows:

$$\mu(U_3(q)) = \{p(q + 1), q^2 - 1, q^2 - q + 1\}.$$  

- The maximal subgroups of $SU_3(q)$ collected in [6] (Table 8. 5) are listed as follows.

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>Conditions</th>
<th>Subgroup</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{q^{1+2}} : (q^2 - 1)$</td>
<td>$SU_3(q_0).\left(\frac{q+1}{q_0+1}, 3\right)$</td>
<td>$q = q_0^r, r$ is an odd prime</td>
<td></td>
</tr>
<tr>
<td>$GU_2(q)$</td>
<td>$d \times SO_3(q)$</td>
<td>$q$ is odd, $q \geq 7$</td>
<td></td>
</tr>
<tr>
<td>$(q + 1)^2 : S_4$</td>
<td>$q \neq 5$</td>
<td>$3_{+}^{1+2} : Q_8.\left(\frac{q+1}{3}, 9\right)$</td>
<td>$p = q \equiv 2 \pmod{3}, q \geq 11$</td>
</tr>
<tr>
<td>$(q^2 - q + 1) : 3$</td>
<td>$q \neq 3, 5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

According to the notation of [6], $d = |Z(SU_3(q))| = (3, q + 1)$. The cyclic group of order $n$ is denoted by $n$. An elementary abelian group of order $p^n$ is denoted by $E_{p^n}$ or just by $p^n$. For a prime $p$, $p^{1+2n}$ or $p^{1+2n}$ is used for the particular case of an extraspecial group. For each prime number $p$ and positive $n$, there are just two types of extraspecial group, which are central products of $n$ non-abelian groups of order $p^3$. For an odd prime $p$, the subscript is $+$ or $-$ according as the group has exponent $p$. 


or \( p^2 \). For elementary abelian groups \( A \), we write \( A^{m+n} \) to mean a group with an elementary abelian normal subgroup \( A^m \) such that the quotient is isomorphic to \( A^n \).

For two groups \( A \) and \( B \), a split extension (resp. a non-split extension) is denoted by \( A : B \) (resp. \( A.B \)). Moreover, \( A \times B \) denotes the direct product of \( A \) and \( B \). (See \([8]\))

Using the information above, the compact form of \( \Gamma_s(U_3(q)) \) with \( q = 3^f \) is found in Figure 1.

![Diagram](image)

**Fig. 1.** \( \Gamma_s(U_3(q)), \ q = 3^f \)

### 3. Characterization of Some Projective Special Unitary Groups

In this section, we will show that the projective special unitary groups \( U_3(q) \) defined over a field of characteristic 3 with some conditions, are completely characterized by their orders and degree pattern of solvable graphs.

First of all, we give a terminology which was introduced in \([1]\). Let \( m \) be a positive integer with the following factorization into distinct prime power factors \( m = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k} \) for some positive integers \( n_i \) and \( k \). Then we set

\[
\text{mpf}(m) := \max\{p_i^{n_i} \mid 1 \leq i \leq k\}.
\]

For convenience, we bring \( \text{mpf}(|S|) \) in Tables 1 and 2 where \( S \) is a sporadic simple group or a simple group of Lie type.

**Theorem 3.1.** All simple groups \( U_3(q) \) where \( q = 3^n \) (\( n \geq 2 \)), such that \( |\pi(q^2 - q + 1)| = 1 \) are \( \text{OD}_s \)-characterizable.

**Proof.** Let \( G \) be a finite group satisfying

\[
|G| = |U_3(q)| = q^3(q^2 - 1)(q^2 + 1),
\]

and \( D_s(G) = D_s(U_3(q)) \), where \( q = 3^f \) and \( |\pi(q^2 - q + 1)| = 1 \). In what follows, we prove that \( G \cong U_3(q) \).

The solvable graph of \( U_3(q) \) with \( q = 3^f \) is shown in Figure 1. Considering Figure 1, we observe that

\[
d_s(3) = |\pi(G)| - 1.
\]
Assume that \( \pi(q^2 - q + 1) = \{p\} \) where \( p \) is a prime. According to the degree pattern of solvable graph of \( G \), \( d_s(p) = 1 \). Hence it is seen from Lemma 2.11 that \( \tilde{\Gamma}(G) = (\Gamma_s(G) - \{3\})^c \) is connected and any disjoint pair of vertices of \( \Gamma_s(G) \) can be expressed by only one GKS-series, say \( 1 \rightarrow M \rightarrow E \rightarrow G \rightarrow N \), such that \( M \) and \( G/N \) are 3-groups. Furthermore, using the structure of the degree pattern of the solvable graph of \( G \), we can get that 3 is adjacent to \( p \). It is also found from Lemma 2.9 that \( p \in \pi(N/M) \). Let \( |M| = 3^m \) and \( |G/N| = 3^n \). Thus we can conclude that

\[
|N/M| = 3^{3f - m - n - 1}(q^2 - 1)(q^3 + 1),
\]

where \( N/M \) is a non-abelian simple group.

So by the classification of finite simple groups, the possibilities for \( N/M \) are as follows:

- an alternating group \( A_k \) on \( k \geq 5 \) letters;
- one of the 26 sporadic simple groups;
- a simple group of Lie type.

If \( N/M \cong U_3(q) \), then \( M = 1, N = G \) and thus \( G \cong U_3(q) \), as required. Therefore, we may suppose that \( N/M \) is isomorphic to the non-abelian simple group \( S \cong U_3(q) \) and try to get a contradiction.

In the rest of proof we will use the following facts. The solvable graph of \( N/M \) is a subgraph of the solvable graph of \( G \) which follows that \( d_s(p) = 1 \) in the solvable graph of \( N/M \) because \( \Gamma_s(N/M) \) is connected.

Moreover, we have

\[
\text{mpf}(|S|) = \text{mpf}(|N/M|).
\]

Hence we first compute the value \( \text{mpf}(3^{3f - m - n - 1}(3^{2f} - 1)(3^{3f} + 1)) \).

It is good to note that \( 3^{f - 1} < 3^f \), \( \text{mpf}((3^f + 1)^2) < 3^{2f} \) and \( 3^f < 3^{2f} - 3f + 1 = p \). So we can conclude that \( \text{mpf}(|N/M|) = p \) or \( 3^l \) where \( l \) is a natural number.

(1) \( S \) is not isomorphic to an alternating group \( A_k \), \( k \geq 5 \).

Suppose that \( S \) is isomorphic to an alternating group \( A_k \), \( k \geq 5 \). Since \( d_s(p) = 1 \), thus it is seen from the spectrum of alternating groups obtained in [11] that \( k \leq p + 3 \). On the other hand, we have

\[
k! = |A_k| = |S| = |N/M| = 3^{3f - m - n - 1}(3^{2f} - 1)(3^{3f} + 1) = 3^{2f - m - n - 1}(p - 1)p(p + 3^{f + 1}),
\]

which is a contradiction.

(2) \( S \) is not isomorphic to one of the 26 sporadic simple groups.

Assume that \( S \) is isomorphic to one of the 26 sporadic simple groups. As mentioned before, \( \text{mpf}(|S|) = \text{mpf}(|N/M|) = p \) or \( 3^l \) where \( l \) is a natural number. Hence, according to Table
1, the possibilities for $S$ are: $M^cL, J_1, J_3, Co_3, Fi_{23}, Fi_{24}, Th$. For convenience, we collect the orders and degree pattern of solvable graphs of these groups in Table 3. (see Table 1 in [3])

We recall that the solvable graph of $N/M$ has a vertex of degree 1. Therefore, $S \neq J_1$. Assume that $S \cong M^cL$. It is seen that $p = 11$ and so $3^f(3^f - 1) = 10$ that is impossible. Other groups may be verified similarly.

(3) $S$ is not isomorphic to a simple group of Lie type, except $U_3(q)$.

Let $dL_n(q)$ be a finite simple group of Lie type of rank $n$, defined over the finite field $K$ of order $q = p^f$. We can observe that $dL_m(q)$ contains an isomorphic copy of $dL_n(q)$ where $m \geq n$. So we conclude that $\Gamma_s(dL_n(q))$ is a subgraph of $\Gamma_s(dL_m(q))$. It follows that in the case when $d_s(r) \geq 2$ in $\Gamma_s(dL_n(q))$ for any prime $r \in \pi(dL_n(q))$, then $S$ is not isomorphic to $dL_m(q)$ where $m \geq n$.

We only discuss on some of these cases, for example, we consider the cases $L_n(q_0), C_n(q_0), 2G_2(q_0)$. Other cases are similar, thus we omit them.

- Suppose that $S$ is isomorphic to $L_n(q_0)$ for some integer $n \geq 2$ and a power $q_0$ of a prime $p_0$. Then, we have

$$|S| = |L_n(q_0)| = (n, q_0 - 1)^{-1} \cdot q_0^{n(n-1)/2} \prod_{i=2}^{n} (q_0^i - 1).$$

Using Table 2,

$$\text{mpf}(|A_n(q_0)|) = q_0^{n(n-1)/2} \quad (n \geq 2),$$

and hence $q_0^{n(n-1)/2} = \text{mpf}(|L_n(q_0)|) = \text{mpf}(|S|) = \text{mpf}(|N/M|)$. If $\text{mpf}(|N/M|) = p$, then $q_0 = p_0 = p$ and $n(n-1)/2 = 1$, which is a contradiction. So we may assume that $\text{mpf}(|N/M|) = 3^f$ for a natural number $l$, which implies that $q_0$ is a power of 3.

First consider $L_2(q_0)$. In [3], $\Gamma_s(L_2(q_0))$ for the odd number $q_0$, was obtained as follows:

![Fig. 2. $\Gamma_s(L_2(q_0))$, $5 < q_0 \equiv -1 \pmod{4}$.](image)

![Fig. 3. $\Gamma_s(L_2(q_0))$, $5 \leq q_0 \equiv 1 \pmod{4}$.](image)

Let $S$ be isomorphic to $L_2(q_0)$. Since $\Gamma_s(S)$ has a vertex of degree 1, it forces that $|\pi(\frac{q_0+1}{2})| \leq 2$. Now, we can conclude from Lemma 2.12 that $q_0 = 9$ or 81. Then $p = 5$ or 41 which is a contradiction because in these cases $p \neq q^2 - q + 1$.

Suppose now that $S$ is isomorphic to $L_3(9)$ or $L_3(81)$. Then according to the compact form of $\Gamma_s(L_3(q_0))$ found in [3], we get that $|\pi(q_0^2 + q_0 + 1)| = 1$ that is not true.
• Assume that $S$ is isomorphic to $C_n(q_0) = \text{PSp}_{2n}(q_0)$ for some integer $n \geq 2$ and a power $q_0$ of a prime $p_0$. As a similar argument to above, we may suppose that $\text{mpf}(|N/M|) = 3^l$ for a natural number $l$, which implies that $q_0$ is a power of 3.

According to the spectrum of symplectic groups obtained in [7] and the maximal subgroups $\text{Sp}_4(q_0)$ (Table 8. 12 in [6]), we find that the solvable graph of $\text{PSp}_4(q_0)$ where $q_0$ is a power of 3 is as follows:

\[
\begin{array}{c}
\pi(q_0-1) \setminus \{2\} & \xrightarrow{3} & \pi(q_0+1) \setminus \{2\} \\
& \xrightarrow{2} & \pi(q_0^2) \setminus \{2\}
\end{array}
\]

**Fig. 4.** $\Gamma_s(\text{PSp}_4(q_0)), q_0 = 3^k$.

Since $\Gamma_s(S)$ has a vertex of degree 1, it forces that $|\pi(q_0^2 + 1)| = 2$. It is enough to consider $q_0^2$ instead of $q$ in Lemma 2.12 and conclude that $q_0 = 9$. Then $p = 41$ which is a contradiction because in this case $p \neq q^2 - q + 1$.

Let $S$ be isomorphic to $\text{PSp}_6(9)$. We found from the spectrum of symplectic groups that for every prime $r \in \pi(S) \setminus \{73\}$, $d_s(r) > 1$. Moreover, if $73 = q^2 - q + 1$, then $q = 9$. It follows that $|\text{PSp}_6(9)| = |U_3(9)|$ which is a contradiction.

• Suppose that $S$ is isomorphic to Ree groups $^{2}G_2(q_0)$ where $q_0 = 3^{2k+1}$, for $k \geq 1$.

According to the spectrum of $^{2}G_2(q_0)$ (see [5, Lemma 4]) and the list of maximal subgroups of $^{2}G_2(q_0)$ in [9], we can obtain that the solvable graph of $^{2}G_2(q_0)$ is as follows:

\[
\begin{array}{c}
\pi(q_0 + \sqrt{3q_0} + 1) & \xrightarrow{3} & \pi(q_0 + \sqrt{3q_0} - 1) \\
& \xrightarrow{2} & \pi(q_0 + \frac{\sqrt{3q_0} - 1}{2}) \setminus \{2, 3\}
\end{array}
\]

\[
\begin{array}{c}
\pi(q_0 + \sqrt{3q_0} + 1) & \xrightarrow{3} & \pi(q_0 + \sqrt{3q_0} - 1) \\
& \xrightarrow{2} & \pi(q_0 + \frac{\sqrt{3q_0} - 1}{2}) \setminus \{2, 3\}
\end{array}
\]

**Fig. 5.** $\Gamma_s(^{2}G_2(q_0)), q = 3^{2k+1} > 3$.

Assume that $\Gamma_s(^{2}G_2(q_0))$ has a vertex of degree 1. By an easy computation, we get from Lemma 2.12 that $|\pi(q_0 + \sqrt{3q_0} + 1)| \neq 2$ and $|\pi(q_0 + \sqrt{3q_0} - 1)| \neq 1$. If $|\pi(q_0 + \sqrt{3q_0} + 1)| = 1$, then $q_0 + \sqrt{3q_0} = q^2 - q$. It implies that $3^{k+1}(3^k + 1) = 3^f(3^f - 1)$ that is impossible. Using a similar argument, $|\pi(q_0 - \sqrt{3q_0} + 1)| \neq 1$.

This completes the proof of theorem. \(\square\)
**Table 1. The order and mpf of a sporadic simple group S.**

| $S$  | $|S|$      | mpf($|S|$) | $S$  | $|S|$      | mpf($|S|$) |
|------|------------|-----------|------|------------|-----------|
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | $2^7$ | $Co_2$ | $2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ | $2^{18}$ |
| $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | $2^4$ | $Fi_{23}$ | $2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ | $3^{13}$ |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | $2^6$ | $Co_1$ | $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$ | $2^{21}$ |
| $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | $2^7$ | $Ru$ | $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ | $2^{14}$ |
| $HS$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | $2^9$ | $Fi'_{24}$ | $2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 31$ | $3^{16}$ |
| $McL$ | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | $3^6$ | | | $23 \cdot 29$ |
| $Suz$ | $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | $2^{13}$ | $O'N$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ | $2^9$ |
| $Fi_{22}$ | $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | $2^{17}$ | $Th$ | $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ | $3^{10}$ |
| $He$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 11$ | $2^{10}$ | $J_4$ | $2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31$ | $2^{21}$ |
| $J_1$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ | $B$ | $2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | $2^{41}$ |
| $J_3$ | $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ | $2^7$ | | | $19 \cdot 23 \cdot 31 \cdot 47$ |
| $HN$ | $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$ | $2^{14}$ | $Ly$ | $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ | $5^6$ |
| $M_{23}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ | $2^7$ | | | $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ | $5^6$ |
| $M_{24}$ | $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ | $2^{10}$ | $M$ | $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17$ | $2^{46}$ |
| $Co_3$ | $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ | $2^{10}$ | | | $19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ | $7^{10}$ |
Table 2. The order and $mpf$ of a simple group of Lie type $S$.

| $S$          | Restrictions on $S$ | $|S|$                                                                 | $mpf(|S|)$                        |
|--------------|---------------------|----------------------------------------------------------------------|-----------------------------------|
| $L_{n+1}(q)$ | $n \geq 2$          | $(n + 1, q - 1)^{-1}q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - 1)$        | $q^{n(n+1)/2}$                    |
| $L_2(q)$     | $|\pi(q + 1)| = 1$  | $(2, q - 1)^{-1}q(q - 1)(q + 1)$                                     | $q + 1$                          |
| $L_2(q)$     | $|\pi(q + 1)| \geq 2$| $(2, q - 1)^{-1}q(q - 1)(q + 1)$                                     | $q$                              |
| $B_n(q)$     | $n \geq 2$          | $(2, q - 1)^{-1}q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$                | $q^{n^2}$                        |
| $C_n(q)$     | $n \geq 2$          | $(2, q - 1)^{-1}q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$                | $q^{n^2}$                        |
| $D_n(q)$     | $n \geq 4$          | $(4, q^n - 1)^{-1}q^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$ | $q^{n(n-1)}$                     |
| $G_2(q)$     |                     | $q^6(q^6 - 1)(q^2 - 1)$                                               | $q^6$                            |
| $F_4(q)$     |                     | $q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$                | $q^{24}$                         |
| $E_6(q)$     |                     | $(3, q - 1)^{-1}q^{12}(q^9 - 1)(q^5 - 1)|F_4(q)|$                  | $q^{36}$                         |
| $E_7(q)$     |                     | $(2, q - 1)^{-1}q^{30}(q^{18} - 1)(q^{14} - 1)(q^{10} - 1)|F_4(q)|$ | $q^{63}$                         |
| $E_8(q)$     |                     | $q^{36}(q^{30} - 1)(q^{12} + 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)$| $q^{120}$                        |
| $U_{n+1}(q)$ | $(n, q) \neq (2, 3), (3, 2)$ | $(n + 1, q + 1)^{-1}q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - (-1)^i)$ | $q^{n(n+1)/2}$                    |
| $n \geq 2$  |                     |                                                                      |                                   |
| $U_4(2)$     |                     | $2^6 \cdot 3^4 \cdot 5$                                              | $3^4$                            |
| $U_3(3)$     |                     | $2^5 \cdot 3^3 \cdot 7$                                              | $2^5$                            |
| $2B_2(q)$    | $q = 2^{2m+1}$       | $q^2(q^2 + 1)(q - 1)$                                                 | $q^2$                            |
| $|\pi(q^2 + 1)| \geq 2$ |                     |                                                                      |                                   |
| $2B_2(q)$    | $q = 2^{2m+1}$       | $q^2(q^2 + 1)(q - 1)$                                                 | $q^2 + 1$                        |
| $|\pi(q^2 + 1)| = 1$ |                     |                                                                      |                                   |
| $2D_n(q)$    | $n \geq 4$          | $(4, q^n + 1)^{-1}q^{n(n-1)}(q^n + 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$ | $q^{n(n-1)}$                     |
| $3D_4(q)$    |                     | $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$                            | $q^{12}$                         |
| $2G_2(q)$    | $q = 3^{2m+1}$       | $q^3(q^3 + 1)(q - 1)$                                                 | $q^3$                            |
| $2F_4(q)$    | $q = 2^{2m+1}$       | $q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$                            | $q^{12}$                         |
| $2E_6(q)$    |                     | $(3, q + 1)^{-1}q^{12}(q^9 + 1)(q^5 + 1)|F_4(q)|$                    | $q^{36}$                         |
Table 3.

| $S$   | $|S|$                | $D_8(S)$          |
|-------|---------------------|-------------------|
| $M^eL$| $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | $(3,3,3,2,1)$    |
| $J_1$ | $2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ | $(5,4,3,2,2,2)$  |
| $J_3$ | $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ | $(3,3,2,1,1)$    |
| $Co_3$| $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ | $(4,3,3,2,3,1)$  |
| $Fi_{23}$| $2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ | $(6,4,4,3,3,2,1,1)$ |
| $Fi'_{24}$| $2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ | $(7,5,4,4,4,2,1,1,2)$ |
| $Th$  | $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ | $(4,5,3,2,2,1,1)$ |

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