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THE EXISTENCE TOTALLY REFLEXIVE COVERS

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ABSTRACT. Let R be a commutative Noetherian ring. We prove that over a local ring R every finitely generated R-module M of finite Gorenstein projective dimension has a Gorenstein projective cover $\varphi : C \to M$ such that C is finitely generated and the projective dimension of $\operatorname{Ker}\varphi$ is finite and φ is surjective.

1. Introduction

Throughout this paper $(R, \mathfrak{m})$ is Noetherian local ring with residue field k. In order to study the structure of a module, we may try to approximate the module using the so-called $\mathcal{F}$-cover where $\mathcal{F}$ is a class of R–modules. The crucial question is the existence covers. Also precovering and covering classes play a fundamental role in relative homological algebra. They are used to construct resolutions, minimal resolutions and to compute relative derived functors. A fundamental question in module theory is the existence of such covers. We consider the existence of totally reflexive covers.

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We recall from [3], that a module G is Gorenstein projective if there is an exact sequence

$$... \to P_2 \to P_1 \to P_0 \to P^0 \to P^1 \to P^2 \to ...$$

of projective modules such that $G = \mathrm{Ker}\,(P^0 \to P^1)$ and such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact when P is projective module. If all projective modules are chosen finitely generated, then G is called totally reflexive.

The Gorenstein projective dimension of modules are defined in the obvious way. The Gorenstein projective dimension of a finitely generated module over a Noetherian ring is just the G-dimension of that module (in the sense of Auslander cf.[1, Theorem: 4.2.6]). So, every finitely generated Gorenstein projective module is totally reflexive.

Let $\mathcal{F}$ be a class of modules. Then an $\mathcal{F}$ -precover of a module M is a homomorphism $\varphi : F \to M$ with $F \in \mathcal{F}$ such that $\mathrm{Hom}_R(G, F) \to \mathrm{Hom}_R(G, M) \to 0$ is exact for all $G \in \mathcal{F}$. If, moreover, any endomorphism $f : F \to F$ such that $\varphi f = \varphi$ is an automorphism, then $\varphi : F \to M$ is said to be an $\mathcal{F}$-cover.

Auslander announced that every finitely generated module over a local Gorenstein ring has a minimal Cohen-Macaulay approximation. In fact, Auslander's Theorem asserts that finitely generated modules over a such ring have Gorenstein projective covers. The main result of [4], extended Auslander's Theorem. It is proved that all finitely generated modules of finite Gorenstein projective dimension over a Cohen-Macaulay local ring admitting a dualizing module, have Gorenstein projective cover (see [4, Theorem 5.5]).

In this conection, we extend the main result of [4] by proving that over a local ring all finitely generated R-modules M of finite Gorenstein projective dimension have Gorenstein projective cover $\varphi : C \to M$ where C is finitely generated such that projective dimension of $\mathrm{Ker}\,\varphi$ is finite and φ is surjective. Since C is finitely generated, $\varphi : C \to M$ is totally reflexive cover. Note that this result recover Auslander's Theorem because if R is Gorenstein then all finitely generated modules have finite Gorenstein projective dimension.

In [4] Enochs, Jenda and Xu discussed whether the condition that M has finite Gorenstein projective dimension in the above result is necessary or not. By [2, Corollary 3.6], every finitely generated R–module has totally reflexive cover if and only if R is Gorenstein or every totally reflexive R–module is free. This result shows that the assumption of finiteness of the Gorenstein projective dimension is necessary.

2. The existence of totally reflexive covers of modules

In this section $(R, \mathfrak{m})$ is a local ring. In the following we review a result on totaly reflexive precovers:

Theorem 2.1. [5, Corollary 2.13] *Every finitely generated R–module N with finite Gorenstein projective dimension has a surjective Gorenstein projective precover, $0 \to K \to C \to N \to 0$, such that C is finitely generated R–module and the kernel K has finite projective dimension.*

If R is Henselian (e.g. complete) and the class $\mathcal{F}$ is closed under direct summands, then a finitely generated R–module admits an $\mathcal{F}$-cover if and only if it admits an $\mathcal{F}$-precover ([6, Corollary 2.5]). On the other hand the category of Gorenstein projective R–modules is closed under direct summands (cf.[5, Theorem 2.5]). Therefore by Theorem 2.1, we have the next Corollary:

Corollary 2.2. *If R is local and Henselian, then every finitely generated R–module of finite Gorenstein projective dimension has a Gorenstein projective cover.*

Theorem 2.6 is our main result in this section. To prove it, we need a couple of results. We recall that the projective dimension of an R–module M is denoted by $\text{proj.dim}_R M$.

Proposition 2.3. *Let M be an R–module of finite Gorenstein projective dimension and $\varphi : C \to M$ be a surjective R–homomorphism, where C is Gorenstein projective. Suppose that M has a Gorenstein projective cover. If $\text{proj.dim}_R \text{Ker}\,\varphi < \infty$ and $\text{Ker}\,\varphi$ contains no nonzero projective direct summands of C, then $\varphi : C \to M$ is a Gorenstein projective cover of M.*

Proof. Assume that the conditions hold and $\psi : D \to M$ is a Gorenstein projective cover. We have the following commutative diagram:

$$\begin{array}{ccc} D & & \\ f \downarrow & \overset{\psi}{\searrow} & \\ C & \xrightarrow{\varphi} & M \\ g \downarrow & \underset{\psi}{\nearrow} & \\ D & & \end{array}$$

Since ψ is a cover, gf is an automorphism of D. So, $\text{Ker}\, g$ is a direct summand of C contained $\text{Ker}\,\varphi$. So $\text{Ker}\, g$ is Gorenstein projective. But $\text{Ker}\, g$ is also isomorphic to a direct summand of $\text{Ker}\,\varphi$ and so $\text{proj.dim}_R \text{Ker}\, g < \infty$. Hence, by [3, Proposition 10.2.3], $\text{Ker}\, g$ is projective and so by assumption is 0. Hence, g is an isomorphism, and so $\varphi : C \to M$ is a Gorenstein projective cover. □

Proposition 2.4. *(i) For any finitely generated R–module M of finite Gorenstein projective dimension, there exists a Gorenstein projective precover $\varphi : C \to M$ with C is finitely generated, proj.dim$_R$Ker$\varphi < \infty$ and such that C has a direct sum decomposition $C = U \oplus F$ such that U has no nonzero free direct summands and such that $F \to M/\varphi(U)$ is a projective cover.*

(ii) If $\varphi : C \to M$ is a homomorphism and $C = U \oplus F$ where U has no nonzero free direct summands and $F \to M/\varphi(U)$ is a projective cover, then Kerφ contains no nonzero free direct summands of C.

Proof. (i) By Theorem 2.1, there is a Gorenstein projective precover $\psi : D \to M$ such that proj.dim$_R$Ker$\,\psi < \infty$ and D is finitely generated. We can choose $D = U \oplus \bar{F}$, where $\bar{F}$ is free and U has no nonzero free direct summand. But ψ is surjective, so $\bar{\psi} : \bar{F} \to M/\psi(U)$ is surjective. Let $\theta : F \to M/\psi(U)$ be projective cover. Therefore there exists an R–epimorphism $\alpha : \bar{F} \to F$ such that $\theta\alpha = \bar{\psi}$. We have $\bar{F} = F \oplus \mathrm{Ker}\,\alpha$. Set $\mathrm{Ker}\,\alpha = \widetilde{F}$. Then $\widetilde{F}$ is free and the map $\widetilde{F} \to M/\psi(U)$ is 0. Hence $\psi(\widetilde{F}) \subseteq \psi(U)$. Since $\widetilde{F}$ is free, so there exists an R–homomorphism $g : \widetilde{F} \to U$ such that $(\psi|_U)g = \psi|_{\widetilde{F}}$. Now, let $C = U \oplus F$ and $\varphi = \psi\,|_C$. Thus, there exists an R–homomorphism from D to C, where is given by:

$$\beta = \begin{pmatrix} id_U & 0 & 0 \\ 0 & id_F & 0 \\ 0 & 0 & g \end{pmatrix}$$

It is easy to see that $\beta(x) = x$ for all $x \in C$, and $\psi\beta(x) = \psi(x)$ for all $x \in D$. Hence $\mathrm{Ker}\,\varphi$ is direct summand of $\mathrm{Ker}\,\psi$, and so proj.dim$_R$Ker$\,\varphi < \infty$.

(ii) Now by contradiction, assume that $x = x_1 + x_2 \in U \oplus F$ where generates a free direct summand of C and $x \in \mathrm{Ker}\,\varphi$. Then $\varphi(x_1) = \varphi(-x_2) \in \varphi(U)$. Since $\bar{\psi}$, is a projective cover and maps $-x_2$ to 0, so $-x_2 \in \mathfrak{m}F$. But x generates free direct summand of C, so there exists $\delta : C \to R$, such that $\delta(x) = 1$. Then $\delta(x_1) + \delta(x_2) = 1$. Since $x_2 \in \mathfrak{m}F$, so $\delta(x_2) \in \mathfrak{m}$ and $\delta(x_1)$ is a unit of R. Hence U has a free direct summand of rank 1. This contradicts our assumption and the proof is complete. □

In the proof of [4, Lemma 5.4], there is no use of the assumptions that R is Cohen-Macaulay and it admits dualizing module. So, we have:

Lemma 2.5. *Let $\hat{R}$ denote the completion of R respect to $\mathfrak{m}$–adic topology. For a finitely generated R–module M, we let $\hat{M}$ denote the completion of M. For any finitely generated R–modules M and N, N is isomorphic to a direct summand of M if and only if $\hat{N}$ is isomorphic (as an $\hat{R}$) to a direct summand of $\hat{M}$.*

Theorem 2.6. *Let M be a finitely generated R-module of finite Gorenstein projective dimension. Then M has a (necessarily surjective) Gorenstein projective cover $\varphi : C \to M$. Furthermore, C is finitely generated and $\text{proj.dim}_R \text{Ker}\,\varphi < \infty$.*

Proof. By Proposition 2.4(i), there is a Gorenstein projective precover $\varphi : C \to M$ such that C is finitely generated and $\text{proj.dim}_R \text{Ker}\,\varphi < \infty$ and such that C has a direct sum decomposition $C = U \oplus F$ where U has no nonzero free summands and such that $\widetilde{\varphi} : F \to M/\varphi(U)$ is a projective cover. So, we have the $\hat{R}$-homomorphism (necessarily surjective) $\hat{\varphi} : \hat{C} \to \hat{M}$, with $\text{proj.dim}_{\hat{R}} \text{Ker}\,\hat{\varphi} < \infty$, and $\hat{C} = \hat{U} \oplus \hat{F}$ such that by Lemma 2.5, $\hat{U}$ has no nonzero free summands. Since $\text{Ker}\,\widetilde{\varphi} \subseteq \mathfrak{m}F$, we have $\text{Ker}\,\hat{\widetilde{\varphi}} \subseteq \hat{\mathfrak{m}}\hat{F}$. Therefore $\hat{\widetilde{\varphi}} : \hat{F} \to \widehat{M/\varphi(U)}$ is projective cover. Hence $\widetilde{\hat{\varphi}} : \hat{F} \to \hat{M}/\hat{\varphi}(\hat{U})$ is projective cover.

We deduce from the above facts and Proposition 2.4(ii) that $\text{Ker}\,\hat{\varphi}$ contains no nonzero free direct summands of $\hat{C}$. On the other hand we know that $\hat{M}$ is a finitely generated $\hat{R}$–module of finite Gorenstein projective dimension, and so by Corollary 2.2, $\hat{M}$ has a Gorenstein projective cover. Hence by Proposition 2.3, $\hat{\varphi} : \hat{C} \to \hat{M}$ is a cover.

Now, let $f : C \to C$ be such that $\varphi f = \varphi$. Then $\hat{\varphi}\hat{f} = \hat{\varphi}$ and consequently $\hat{f}$ is an automorphism of $\hat{C}$, because $\hat{\varphi} : \hat{C} \to \hat{M}$ is a cover. This implies that f is automorphism of C, and so that $\varphi : C \to M$ is our desired Gorenstein projective cover. □

One can see that any Gorenstein projective cover $\varphi : C \to M$, which C is finitely generated is in fact, a totally reflexive cover. So, we can end the paper with the following corollary:

Corollary 2.7. *Let M be a finitely generated R-module of finite Gorenstein projective dimension. Then M has a (necessarily surjective) totally reflexive cover $\varphi : C \to M$ with $\text{proj.dim}_R \text{Ker}\,\varphi < \infty$.*

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