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SOME REMARKS ON GENERALIZATIONS OF CLASSICAL PRIME SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. Suppose that $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of M is called an $(n-1,n)$-φ-classical prime submodule, if whenever $r_1, \dots, r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1} m \in N \setminus \varphi(N)$, then $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$, for some $i \in \{1, \dots, n-1\}$ $(n \geqslant 3)$. In this work, $(n-1,n)$-φ-classical prime submodules are studied and some results are established.

1. Introduction

Throughout the paper, all rings are commutative with identity and all modules are unitary. Generalizations of prime submodule play an important role in extending prime submodule. Anderson and Bataineh [5] introduced various generalizations of prime ideals. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. A proper ideal I of R is a ψ-prime ideal, if $ab \in I \setminus \psi(I)$ whenever $a, b \in R$, then $a \in I$ or $b \in I$. Zamani [24] used this

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concept to φ-prime submodule, i.e, a proper submodule N of M is a φ-prime submodule if $rm \in N \setminus \varphi(N)$ whenever $r \in R$ and $m \in M$, then $m \in N$ or $r \in (N : M)$, where $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and $S(M)$ is the set of all submodules of M. Ebrahimpour and Nekooei [13] defined $(n-1,n)$-φ-prime submodule and $(n-1,n)$-ψ-prime ideal and obtained a number of results concerning $(n-1,n)$-φ-prime submodules. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. A proper ideal I of R is $(n-1,n)$-ψ-prime if $r_1 \ldots r_n \in I \setminus \varphi(I)$ whenever $r_1, \ldots, r_n \in R$, then $r_1 \ldots r_{i-1} r_{i+1} \ldots r_n \in I$ for some $i \in \{1, \ldots, n\}$. This concept of $(n-1,n)$-φ-prime submodule has been studied similar to what has been done for $(n-1,n)$-φ-prime submodule. Mostafanasab et. al. [20] defined φ-classical prime submodule. Let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of M is a φ-classical prime submodule of M if $abm \in N \setminus \varphi(N)$ whenever $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$. So taking $\varphi(N) = \emptyset$ (resp. $\varphi(N) = 0$, $\varphi(N) = (N : M)N$, $\varphi(N) = (N : M)^{m-1}N$, $\varphi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$) is a classical prime submodule (resp. weakly classical prime submodule, almost classical prime submodule, m-almost classical prime submodule, ω-classical prime submodule). Some properties of φ-classical prime submodules have been investigated (see[20]). In connection with various definitions of prime submodules and prime ideals, a number of results concerning prime submodules and prime ideals have been estabished in [3], [4], [6], [8], [10], [11], [15], [16], [17] and [19].

In this work, we will continue this concept by a generalization of φ-classical prime submodules. We define an $(n-1,n)$-φ-classical prime submodule of M and investigate a number of results concerning $(n-1,n)$-φ-classical prime submodules. Let M be an R-module and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. Let N be a proper submodule of M and without loss of generality we may assume that $\varphi(N) \subseteq N$. A Submodule N is an $(n-1,n)$-φ-classical prime submodule of M if $r_1 \ldots r_{n-1} m \in N \setminus \varphi(N)$ whenever $r_1, \ldots, r_{n-1} \in R$ and $m \in M$, then $r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1} m \in N$ for some $i \in \{1, \ldots, n-1\}$ $(n \geqslant 3)$. If $\varphi(N) = \emptyset$ (resp. $\varphi(N) = 0$, $\varphi(N) = (N : M)N$, $\varphi(N) = (N : M)^{m-1}N$, $\varphi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$), then a submodule N is called an $(n-1,n)$-classical prime submodule (resp. $(n-1,n)$-weakly classical prime submodule, $(n-1,n)$-almost classical prime submodule, $(n-1,n)$-m-almost classical prime submodule, $(n-1,n)$-ω classical prime submodule). For every two functions φ_1 and $\varphi_2 : S(M) \to S(M) \cup \{\emptyset\}$, we say $\varphi_1 \leqslant \varphi_2$ if $\varphi_1(N) \subseteq \varphi_2(N)$ for each $N \in S(M)$. It is clear that every $(n-1,n)$-φ_1-classical prime submodule is an $(n-1,n)$-φ_2-classical prime submodule of M. Also, if N is an $(n-1,n)$-classical prime submodule of M, then N is an $(n,n+1)$-classical prime submodule of M. Now, let $\psi : R \to S$ be a ring homomorphism and M be an S-module. If N is an $(n-1,n)$-φ-classical prime submodule of S-module M, then N is an $(n-1,n)$-φ-classical prime submodule of R-module M.

2. Results of $(n-1,n)$-φ-Classical Prime Submodules

In the following theorem we give a characterzation of $(n-1,n)$-φ-classical prime submodule ($n \geqslant 3$) similar to what has been done for $(n-1,n)$-φ-prime submodule (see ([13], Theorem 2.6)). Also, this concept has been characterized for φ-classical prime submodules in ([20], Theorem 2.11).

Theorem 2.1. *Let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then the following conditions are equivalent:*
(i) N is an $(n-1,n)$-φ-classical prime submodule.
(ii) For $r_1, \ldots, r_{n-2} \in R$ and $m \in M$ with $r_1 \ldots r_{n-2}m \in M \setminus N$, we have

$$(N : r_1 \ldots r_{n-2}m) = \bigcup_{i=1}^{n-2}(N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m) \cup (\varphi(N) : r_1 \ldots r_{n-2}m)(n \geq 3).$$

Proof. $(i) \Rightarrow (ii)$ Let $r_1 \ldots r_{n-2}m \in M \setminus N$. Suppose that $r \in (N : r_1 \ldots r_{n-2}m)$, so $rr_1 \ldots r_{n-2}m \in N$. If $rr_1 \ldots r_{n-2}m \notin \varphi(N)$, then $rr_1 \ldots r_{n-2}m \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-prime submodule, hence $rr_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m \in N$, for some $i \in \{1, \ldots, n-2\}$. Thus $r \in (N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m)$, for some $i \in \{1, \ldots, n-2\}$. If $rr_1 \ldots r_{n-2}m \in \varphi(N)$, then $r \in (\varphi(N) : r_1 \ldots r_{n-2}m)$. So $(N : r_1 \ldots r_{n-2}m) \subseteq \bigcup_{i=1}^{n-2}(N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m) \cup (\varphi(N) : r_1 \ldots r_{n-2}m)$. Now, since we assume that $\varphi(N) \subseteq N$, the other inclusion always holds.
$(ii) \Rightarrow (i)$ Let $r_1, \ldots, r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N \setminus \varphi(N)$. If $r_1 \ldots r_{n-2}m \in N$, then N is an $(n-1,n)$-φ-classical prime. Now, we can suppose that $r_1 \ldots r_{n-2}m \notin N$. So $(N : r_1 \ldots r_{n-2}m) = \bigcup_{i=1}^{n-2}(N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m) \cup (\varphi(N) : r_1 \ldots r_{n-2}m)$. Since $r_1 \ldots r_{n-1}m \in N$, so $r_{n-1} \in (N : r_1 \ldots r_{n-2}m)$. But $r_{n-1} \notin (\varphi(N) : r_1 \ldots r_{n-2}m)$, because $r_1 \ldots r_{n-2}r_{n-1}m \notin \varphi(N)$. It follows that $r_{n-1} \in \bigcup_{i=1}^{n-2}(N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m)$. Thus $r_{n-1} \in (N : r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}m)$, for some $i \in \{1, \ldots, n-2\}$ and so $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-2}r_{n-1}m \in N$. Thus N is an $(n-1,n)$-φ-classical prime submodule of M. □

We prove the following proposition that is similar to Proposition 2.27 of [20].

Proposition 2.2. *Let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Suppose that $\varphi(N)$ be an $(n-1,n)$-classical prime submodule of M. If N is an $(n-1,n)$-φ-classical prime submodule of M, then N is an $(n-1,n)$-classical prime submodule of M ($n \geqslant 3$).*

Proof. Let $r_1, \ldots, r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N$. If $r_1 \ldots r_{n-1}m \notin \varphi(N)$, then $r_1 \ldots r_{n-1}m \in N \setminus \varphi(N)$. Thus $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$, for some $i \in \{1, \ldots, n-1\}$. Hence N is an $(n-1,n)$-classical prime submodule of M. We consider $r_1 \ldots r_{n-1}m \in \varphi(N)$.

Since $\varphi(N)$ is an $(n-1,n)$-classical prime submodule, $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in \varphi(N) \subseteq N$, for some $i \in \{1,\dots,n-1\}$ and so the proof is complete. □

In the following proposition, we state relations between $(n-1,n)$-φ-classical prime submodules and $(n-1,n)$-φ-prime submodules of M $(n \geqslant 3)$.

Proposition 2.3. *Let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M and $\mathcal{I}(R)$ is the set of all ideals of R.*
(i) If N is a φ-prime submodule of M, then N is an $(n-1,n)$-φ-classical prime submodule of M.
(ii) If N is an $(n-1,n)$-φ-prime submodule of M and $(N : M)$ is an $(n-2,n-1)$-ψ-prime ideal of R with $\psi((N : M)) \subseteq (\varphi(N) : m)$, for each $m \in M$, then N is an $(n-1,n)$-φ-classical prime submodule of M.
(iii) If N is an $(n-1,n)$-φ- prime submodule and $(N : M)$ is an $(n-2,n-1)$-prime ideal of R, then N is an $(n-1,n)$-φ-classical prime submodule of M $(n \geqslant 3)$.

Proof. (i) Let $r_1,\dots,r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1}m \in N \setminus \varphi(N)$. Hence we have $r_1(r_2 \dots r_{n-1}m) \in N \setminus \varphi(N)$. Since N is a φ-prime submodule of M, so $r_1 \in (N : M)$ or $r_2 \dots r_{n-1}m \in N$. If $r_1 \in (N : M)$, then $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$ and from $r_2 \dots r_{n-1}m \in N$, we get that N is an $(n-1,n)$-φ-classical prime submodule of M.
(ii) Let $r_1,\dots,r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1}m \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-prime submodule of M, so $r_1 \dots r_{n-1} \in (N : M)$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$ for some $i \in \{1,\dots,n-1\}$. If $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$, for some $i \in \{1,\dots,n-1\}$, then N is an $(n-1,n)$-φ-classical prime submodule of M. Now, we have $r_1 \dots r_{n-1} \in (N : M)$. Since $r_1 \dots r_{n-1}m \notin \varphi(N)$, so $r_1 \dots r_{n-1} \notin \psi((N : M))$, because of $\psi((N : M)) \subseteq (\varphi(N) : m)$ for each $m \in M$. Thus we have $r_1 \dots r_{n-1} \in (N : M) \setminus \psi((N : M))$. Since $(N : M)$ is an $(n-2,n-1)$-ψ-prime ideal of R, so $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1} \in (N : M)$, for some $i \in \{1,\dots,n-1\}$. Therefore $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$, for some $i \in \{1,\dots,n-1\}$. Consequently, N is an $(n-1,n)$-φ-classical prime submodule.
(iii) It is straightforward. □

The following theorem introduces the relation between ideal $(N : M)$ of R and submodule N of M in the case of generalization.

Theorem 2.4. *Let M be an R-module and N be a proper submodule of M. Also, let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*
(i) If N is an $(n,n+1)$-φ-classical prime submodule of M with $(\varphi(N) : m) \subseteq \psi((N : M))$ for

each $m \in M$, then $(N : M)$ is an $(n-1, n)$-ψ-prime ideal of R $(n \geqslant 3)$.
(ii) If $(N : m)$ is an $(n-2, n-1)$-ψ-prime ideal of R such that $\psi((N : m)) \subseteq (\varphi(N) : m)$ for each $m \in M$, then N is an $(n-1, n)$-φ-classical prime submodule of M.

Proof. (i) Let $r_1, \dots, r_n \in R$ with $r_1 \dots r_n \in (N : M) \setminus \psi((N : M))$. Then $r_1 \dots r_n \in (N : M)$ and $r_1 \dots r_n \notin \psi((N : M))$. So $r_1 \dots r_n M \subseteq N$ and $r_1 \dots r_n \notin (\varphi(N) : m)$ for each $m \in M$, hence $r_1 \dots r_n m \in N \setminus \varphi(N)$ for each $m \in M$. Since N is an $(n, n+1)$-φ-classical prime submodule of M, so $r_1 \dots r_{i-1} r_{i+1} \dots r_n m \in N$ for some $i \in \{1, \dots, n\}$. Therefore $r_1 \dots r_{i-1} r_{i+1} \dots r_n \in (N : M)$, for some $i \in \{1, \dots, n\}$, as required.
(ii) Let $r_1, \dots, r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1} m \in N \setminus \varphi(N)$. Then $r_1 \dots r_{n-1} m \in N$ and $r_1 \dots r_{n-1} m \notin \varphi(N)$. Since $\psi((N : m)) \subseteq (\varphi(N) : m)$, so $r_1 \dots r_{n-1} \notin \psi((N : m))$ and hence $r_1 \dots r_{n-1} \in (N : m) \setminus \psi((N : m))$. Since $(N : m)$ is an $(n-2, n-1)$-ψ-prime ideal of R, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} \in (N : m)$ for some $i \in \{1, \dots, n-1\}$. Thus $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$, as required. □

The following example shows that if N is an $(n-1, n)$-classical prime submodule, then it is not necessarily classical prime.

Example 2.5. Let $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$ where p, q are two distinct prime integers. Notice that $N = \langle (\overline{0}, \overline{0}, 0) \rangle$ is a submodule of $\mathbb{Z}$-module M. We show that N is a $(3, 4)$-classical prime submodule, but is not $(2, 3)$-classical prime. Since $pq(\overline{1}, \overline{1}, 0) \in \langle (\overline{0}, \overline{0}, 0) \rangle$, but $p(\overline{1}, \overline{1}, 0) \neq (\overline{0}, \overline{0}, 0)$ and $q(\overline{1}, \overline{1}, 0) \neq (\overline{0}, \overline{0}, 0)$. Therefore N is not classical prime submodule. Now, we prove that N is a $(3, 4)$-classical prime submodule. Let n_1, n_2, n_3 are nonzero integers and $(\overline{m}, \overline{n}, r) \in \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$ with $n_1 n_2 n_3 (\overline{m}, \overline{n}, r) \in \langle (\overline{0}, \overline{0}, 0) \rangle$. We have $\overline{n_1 n_2 n_3 m} = \overline{0} \in \mathbb{Z}_p$, $\overline{n_1 n_2 n_3 n} = \overline{0} \in \mathbb{Z}_q$ and $r = 0$, so $p | n_1 n_2 n_3 m$ and $q | n_1 n_2 n_3 n$ such that $0 \leqslant m \leqslant p-1$ and $0 \leqslant n \leqslant q-1$. Since $p \nmid m$ and $q \nmid n$, therefore $p | n_1 n_2 n_3$ and $q | n_1 n_2 n_3$. Hence $p | n_i n_j$ and $q | n_i n_j$ where $i \neq j$, $1 \leqslant i, j \leqslant 3$ and so $n_i n_j (\overline{m}, \overline{n}, 0) = (\overline{0}, \overline{0}, 0)$. Consequently N is a $(3, 4)$-classical prime submodule of M.

An R-module M is called a multiplication module if for every submodule N of M, $N = IM$ for some ideal I of R. We have $I = (N : M)$, so $N = (N : M)M$. Let M be a multipilication R-module and K, L be two submodules of M. Then there are ideals I, J of R such that $K = IM$ and $L = JM$ and hence $KL = IJM = IL$. Also, for each $m \in M$ we define $Lm = LRm$, so $Lm = JRm = Jm$. We recall that the N^m is defined by $N^m = (N : M)^m M$, clearly N^m is a submodule of M and $N^m \subseteq N$. For more details concerning multiplication modules and the prime submodules of multiplication modules refer to [2], [14], [23].

Definition 2.6. Suppose that M be a multiplication R-module and $t \geqslant 2$. A proper submodule N of M is said an $(n-1,n)$-t-potent classical prime if whenever $r_1,\ldots,r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N^t$, then $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$ for some $i \in \{1,\ldots,n-1\}$.

Now, we assert the following proposition that is similar to Propostion 2.7 of [20].

Proposition 2.7. *Let M be a multiplication R-module and N be a proper submodule of M. If N is an $(n-1,n)$-m-almost classical prime submodule of M for some $m \geqslant 2$ and N is an $(n-1,n)$-t-potent classical prime submodule of M such that $t \leqslant m$, then N is an $(n-1,n)$-classical prime submodule of M $(n \geqslant 3)$.*

Proof. Suppose that N is an $(n-1,n)$-m-almost classical prime submodule of M. Let $r_1,\ldots,r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N$. Since $N^m \subseteq N^t$, $r_1 \ldots r_{n-1}m \notin N^t$ implies that $r_1 \ldots r_{n-1}m \notin N^m$ and hence $r_1 \ldots r_{n-1}m \in N \setminus N^m$. On the other hand, we have $N \setminus N^m = N \setminus (N:M)^m M = N \setminus (N:M)^{m-1}(N:M)M = N \setminus (N:M)^{m-1}N$. Thus $r_1 \ldots r_{n-1}m \in N \setminus (N:M)^{m-1}N$. Since N is an $(n-1,n)$-m-almost classical prime submodule of M, $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$ for some $i \in \{1,\ldots,n-1\}$. Therefore N is an $(n-1,n)$-classical prime submodule of M. If $r_1 \ldots r_{n-1}m \in N^t$, because N is an $(n-1,n)$-t-potent classical prime submodule of M, then $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$. Therefore N is an $(n-1,n)$-classical prime submodule of M. □

Proposition 2.8. *Let M be a free multiplication R-module and N be a proper submodule of M such that $(N:M)^m$ be a prime ideal of R $(m \geqslant 2)$. If N is an $(n-1,n)$-m-almost classical prime submodule of M, then N is an $(n-1,n)$-classical prime submodule of M.*

Proof. Let $r_1,\ldots,r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N$. If $r_1 \ldots r_{n-1}m \notin (N:M)^{m-1}N$, then $r_1 \ldots r_{n-1}m \in N \setminus (N:M)^{m-1}N$. Since N is an $(n-1,n)$-m-almost classical prime of M and hence $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$, for some $i \in \{1,\ldots,n-1\}$. Thus N is an $(n-1,n)$-classical prime submodule of M. Now suppose that $r_1 \ldots r_{n-1}m \in (N:M)^{m-1}N = (N:M)^m M$ and $\{x_\alpha\}_{\alpha\in\Lambda}$ be a basis for a free R-module M. Thus $m = \sum_{f.s} r'_\alpha x_\alpha$ where $r'_\alpha \in R$ and hence $\sum_{f.s}(r_1 \ldots r_{n-1})r'_\alpha x_\alpha \in (N:M)^m M$. It follows that $r_1 \ldots r_{n-1}r'_\alpha \in (N:M)^m$ for each $\alpha \in \Lambda$. Since $(N:M)^m$ is a prime ideal of R, so $r_1 \ldots r_{n-1} \in (N:M)^m$ or $r'_\alpha \in (N:M)^m$. This shows that $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in (N:M)M = N$ for some $i \in \{1,\ldots,n-1\}$, as required. □

Theorem 2.9. *Let M be a free multiplication R- module and $f: M \to M'$ be an R-module epimorphism. Let N be a proper submodule of M such that $kerf \subseteq N$ and $(N:M)$ be a prime ideal of R. If N is an $(n-1,n)$-φ-classical prime submodule of M, then $f(N)$ is an $(n-1,n)$-classical prime submodule of M'.*

Proof. Let $r_1 \dots r_{n-1}m' \in f(N)$ where $r_1, \dots, r_{n-1} \in R$ and $m' \in M'$. We have $r_1 \dots r_{n-1}f(m) \in f(N)$ where $r_1, \dots, r_{n-1} \in R$ and $m' = f(m) \in M'$ for some $m \in M$. Since $kerf \subseteq N$, so $r_1 \dots r_{n-1}m \in N$. If $r_1 \dots r_{n-1}m \notin \varphi(N)$, then $r_1 \dots r_{n-1}m \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-classical prime submodule of M, so $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$ for some $i \in \{1, ..., n-1\}$ and hence $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m' \in f(N)$ for some $i \in \{1, ..., n-1\}$. Thus $f(N)$ is an $(n-1,n)$-classical prime submodule of M'. Now, assume that $r_1 \dots r_{n-1}m \in \varphi(N)$. Since $\varphi(N) \subseteq N$ and $N = (N : M)M$, so $r_1 \dots r_{n-1}m \in (N : M)M$. Suppose that $\{x_\alpha\}_{\alpha \in \Lambda}$ be a basis for a free R-module M, therefore $m = \sum_{f.s} r'_\alpha x_\alpha$ where $r'_\alpha \in R$. We get $\sum_{f.s} r_1 \dots r_{n-1}r'_\alpha x_\alpha \in (N : M)M$. It follows that $r_1 \dots r_{n-1}r'_\alpha \in (N : M)$. Since $(N : M)$ is a prime ideal of R, so $r_1 \dots r_{n-1} \in (N : M)$ or $r'_\alpha \in (N : M)$. This shows that $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$ for some $i \in \{1, ..., n-1\}$, hence $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m' \in f(N)$ for some $i \in \{1, ..., n-1\}$. So $f(N)$ is an $(n-1,n)$-classical prime submodule of M'. □

Theorem 2.10. *Let M be a free multiplication R-module and N be a proper submodule of M such that $(N : M)$ be a prime ideal of R. If N is an $(n-1,n)$-φ-classical prime submodule of M, then N is an $(n-1,n)$-classical prime submodule of M.*

Proof. It is straightforward. □

The following thorem is stated for φ-classical prime submodules of M (see ([20], Theorem 2.9)). We assert it for $(n-1,n)$-φ-classical prime submodules of M.

Proposition 2.11. *Let $f : M \to M'$ be an R-module epimorphism and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$, $\varphi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*
(i) If N is an $(n-1,n)$-φ-classical prime submodule of M with $kerf \subseteq N$ and $f(\varphi(N)) \subseteq \varphi'(f(N))$, then $f(N)$ is an $(n-1,n)$-φ'-classical prime submodule of M' ($n \geqslant 3$).
(ii) If L is an $(n-1,n)$-φ'-classical prime submodule of M' and $f^{-1}(\varphi'(L)) \subseteq \varphi(f^{-1}(L))$, then $f^{-1}(L)$ is an $(n-1,n)$-φ-classical prime submodule of M ($n \geqslant 3$).

Proof. (i) There exists $m \in M$ such that $f(m) = m'$, $m' \in M'$. Let $r_1, \dots, r_{n-1} \in R$ and $m' \in M'$ with $r_1 \dots r_{n-1}m' \in f(N) \setminus \varphi'(f(N))$. We get $f(r_1 \dots r_{n-1}m) \in f(N)$. Since $kerf \subseteq N$, $r_1 \dots r_{n-1}m \in N$ and since $f(\varphi(N)) \subseteq \varphi'(f(N))$, so $r_1 \dots r_{n-1}m \notin \varphi(N)$. Therefore, $r_1 \dots r_{n-1}m \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-classical prime submodule of M, $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m \in N$, for some $i \in \{1, \dots, n-1\}$. Thus, $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}m' \in f(N)$ for some $i \in \{1, \dots, n-1\}$.
(ii) It is quite similar to (i). □

Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be an R-module where R_i is a commutative ring and M_i is an R_i-module for $i = 1, 2$. Assume that $N = N_1 \times N_2$ be a proper submodule of $M = M_1 \times M_2$ where N_i is a proper submodule of M_i for $i = 1, 2$. Let $\varphi : S(M_1 \times M_2) \to S(M_1 \times M_2) \cup \{\emptyset\}$ and $\varphi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be functions with $\varphi(N_1 \times N_2) = \varphi_1(N_1) \times \varphi_2(N_2)$ for $i = 1, 2$. Now, we prove the following theorems that some properties of this concept have been investigated for φ-classical prime submodules of $M_1 \times M_2$ (see ([20], Theorem 2.36, Theorem 2.39, Theorem 2.40, Theorem 2.41)).

Theorem 2.12. *Let $M_1 \times M_2$ be an $R_1 \times R_2$- module and N_i be a proper submodule of M_i for $i = 1, 2$. If $N_1 \times N_2$ is an $(n-1, n)$-φ-classical prime submodule $M_1 \times M_2$, then N_i is an $(n-1, n)$-φ_i-classical prime submodule of M_i for $i = 1, 2$.*

Proof. Let $i = 1$, $N_1 \neq M_1$, $r_1, \dots, r_{n-1} \in R_1$ and $m_1 \in M_1$ with $r_1 \dots r_{n-1} m_1 \in N_1 \setminus \varphi_1(N_1)$. We have $(r_1 \dots r_{n-1} m_1, 0) \in N_1 \times N_2$ and $(r_1 \dots r_{n-1} m_1, 0) \notin \varphi_1(N_1) \times \varphi_2(N_2)$. So $(r_1 \dots r_{n-1} m_1, 0) = (r_1, 1) \dots (r_{n-1}, 1)(m_1, 0) \in N_1 \times N_2 \setminus \varphi_1(N_1) \times \varphi_2(N_2)$. Since $N_1 \times N_2$ is an $(n-1, n)$-φ-classical prime submodule of $M_1 \times M_2$, so $(r_1, 1) \dots (r_{i-1}, 1)(r_{i+1}, 1) \dots (r_{n-1}, 1)(m_1, 0) \in N_1 \times N_2$ for some $i \in \{1, ..., n-1\}$ and hence $(r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m_1, 0) \in N_1 \times N_2$. Thus $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N_1$. Therefore N_1 is an $(n-1, n)$-φ_1-classical prime submodule of M_1. □

Theorem 2.13. *Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and $N_1 \times N_2$ be a submodule of $M_1 \times M_2$ with $N_i \neq M_i$ for $i = 1, 2$. If N_1 and N_2 are $(n-1, n)$-classical prime submodule of M_1, M_2, resp., then $N_1 \times N_2$ is an $(n-1, n)$-classical prime submodule of $M_1 \times M_2$.*

Proof. Let $(r_1, r'_1), ..., (r_{n-1}, r'_{n-1}) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ with $(r_1, r'_1)...(r_{n-1}, r'_{n-1})(m_1, m_2) \in N_1 \times N_2$. We get $r_1...r_{n-1} m_1 \in N_1$ and $r'_1...r'_{n-1} m_2 \in N_2$. Since N_i is an $(n-1, n)$- classical prime submodule of M_i for $i = 1, 2$, so $r_1...r_{i-1} r_{i+1}...r_{n-1} m_1 \in N_1$ for some $i \in \{1, ..., n-1\}$ and also, $r'_1...r'_{j-1} r'_{j+1}...r'_{n-1} m_2 \in N_2$ for some $j \in \{1, ..., n-1\}$. If $i = j$, then $(r_1, r'_1)...(r_{i-1}, r'_{i-1})(r_{i+1}, r'_{i+1})...(r_{n-1},\ r'_{n-1})(m_1, m'_2) \in N_1 \times N_2$. Thus $N_1 \times N_2$ is an $(n-1, n)$-classical prime submodule of $M_1 \times M_2$. If $i \neq j$, we have $(r_1, r'_1)...(r_j, r'_i)...(r_{n-1}, r'_{n-1})(m_1, m_2) \in N_1 \times N_2$, as required. □

Corollary 2.14. *Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and $\varphi : S(M_1 \times M_2) \to S(M_1 \times M_2) \cup \{\emptyset\}$ be a function with $\varphi(N_1 \times N_2) = \varphi_1(N_1) \times \varphi_2(N_2)$ where $\varphi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function such that $(\varphi_i(M_i) : M_i) = R_i$ for $i = 1, 2$. If N_i is an $(n-1, n)$-φ_i- classical prime submodule of M_i for $i = 1, 2$, then $N_1 \times M_2$ and $M_1 \times N_2$ are $(n-1, n)$-φ-classcal prime submodule of $M_1 \times M_2$ $(n \geq 3)$.*

Proof. Let $(r_1, r'_1), ..., (r_{n-1}, r'_{n-1}) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ with $(r_1, r'_1)...(r_{n-1}, r'_{n-1})(m_1, m_2) \in N_1 \times M_2 \setminus \varphi(N_1 \times M_2)$. It follows that $(r_1...r_{n-1}m_1, r'_1 ...r'_{n-1}m_2) \in N_1 \times M_2$ and $(r_1...r_{n-1}m_1, r'_1...r'_{n-1}m_2) \notin \varphi_1(N_1) \times \varphi_2(M_2)$. So $r_1...r_{n-1} m_1 \in N_1$ and $r_1...r_{n-1}m_1 \notin \varphi_1(N_1)$, because $r'_1...r'_{n-1}m_2 \in \varphi_2(M_2)$ and hence $r_1...r_{n-1}m_1 \in N_1 \setminus \varphi_1(N_1)$. Since N_1 is an $(n-1,n)$-φ_1-classical prime submodule of M_1, so $r_1...r_{i-1}r_{i+1}...r_{n-1}m_1 \in N_1$ for some $i \in \{1, ..., n-1\}$. Therefore $(r_1, r'_1)...(r_{i-1}, r'_{i+1})...(r_{n-1}, r'_{n-1})(m_1, m_2) \in N_1 \times M_2$. Thus $N_1 \times M_2$ is an $(n-1,n)$-φ-classical prime submodule of $M_1 \times M_2$. □

Let S be a multiplicatively closed subset of R. We cognize that every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Let $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $\varphi_S : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $\varphi_S(S^{-1}N) = S^{-1}\varphi(N)$ and $\varphi_S(S^{-1}N) = \emptyset$ if $\varphi(N) = \emptyset$ where N is a submodule of M. Moreover, let M be an R-module. Then the set of zero-divisors of M is denoted by $Z_R(M)$. Now, we state Theorem 2.28 in [20] for $(n-1,n)$-φ-classical prime submodules.

Theorem 2.15. *Let M be an R-module and S be a multiplicatively closed subset of R such that $S^{-1}(\varphi(N)) \subseteq \varphi_S(S^{-1}N)$.*
(i) If N is an $(n-1,n)$-φ-classical prime submodule of M with $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is an $(n-1,n)$-φ_S-classical prime submodule of $S^{-1}M$.
(ii) If $S^{-1}N$ is an $(n-1,n)$-φ_S-classical prime submodule of $S^{-1}M$ such that $S \cap Z_R(N \setminus \varphi(N)) = \emptyset$ and $S \cap Z_R(M \setminus N) = \emptyset$, then N is an $(n-1,n)$-φ-classical prime submodule of M.

Proof. (i) Let $\frac{r_1}{s_1}, ..., \frac{r_{n-1}}{s_{n-1}} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$ with $\frac{r_1}{s_1}...\frac{r_{n-1}}{s_{n-1}}\frac{m}{t} \in S^{-1}N \setminus \varphi_S(S^{-1}N)$. We have $\frac{r_1...r_{n-1}m}{s_1...s_{n-1}t} \in S^{-1}N$ and $\frac{r_1...r_{n-1}m}{s_1...s_{n-1}t} \notin \varphi_S(S^{-1}N)$. Then there exists $u \in S$ such that $ur_1...r_{n-1}m \in N$ and $ur_1...r_{n-1}m \notin \varphi(N)$, so $r_1...r_{n-1}(um) \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-classical prime submodule of M, hence $r_1...r_{i-1}r_{i+1}... r_{n-1}(um) \in N$ for some $i \in \{1, ..., n-1\}$. This shows that $\frac{ur_1...r_{i-1}r_{i+1}...r_{n-1}m}{us_1...s_{i-1}s_{i+1}...s_{n-1}t} = \frac{r_1}{s_1}...\frac{r_{i-1}}{s_{i-1}}\frac{r_{i+1}}{s_{i+1}}...\frac{r_{n-1}}{s_{n-1}}\frac{m}{t} \in S^{-1}N$. Thus $S^{-1}N$ is an $(n-1,n)$-φ_S-classical prime submodule of $S^{-1}M$.
(ii) See ([20], Theorem 2.28). □

Definition 2.16. Let M be an R-module and N be a submodule of M. Then N is said relatively divisible submodule denoted by RD-submodule, if $rN = N \cap rM$ for each $r \in R$. R-module M is called prime module if $rm = 0$ where $r \in R$ and $m \in M$, then $r \in Ann(M)$ or $m = 0$. Now, we assert the following proposition.

Proposition 2.17. *Let M be a prime R-module and N be a proper submodule of M. If N is an RD-submodule of M with $Ann(M) \subseteq (\varphi(N) : M)$, then N is an $(n-1,n)$-φ-classical prime submodule of M.*

Proof. Let $r_1, \ldots r_{n-1} \in R$ and $m \in M$ with $r_1 \ldots r_{n-1}m \in N \setminus \varphi(N)$. Since N is a *RD*-submodule, so $(r_1 \ldots r_{n-1})M \cap N = (r_1 \ldots r_{n-1})N$ and hence $r_1 \ldots r_{n-1}m \in (r_1 \ldots r_{n-1})M \cap N = (r_1 \ldots r_{n-1})N$. Therefore $r_1 \ldots r_{n-1}m = r_1 \ldots r_{n-1}m'$ for some $m' \in N$ and so $(r_1 \ldots r_{n-1})(m - m') = 0$. Since M is prime, hence $r_1 \ldots r_{n-1} \in Ann(M)$ or $m - m' = 0$. But if $r_1 \ldots r_{n-1} \in Ann(M)$, then $r_1 \ldots r_{n-1} \in (\varphi(N) : M)$, so $r_1 \ldots r_{n-1}m \in \varphi(N)$ which contradicts with our assumption. Thus $m - m' = 0$, hence $m \in N$ and so $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$ for some $i \in \{1, ..., n-1\}$, as required. □

Definition 2.18. [22] A commutative ring R is called a u-ring if for each ideal I of R with $I \subseteq \cup_{i=1}^{n} I_i$, then $I \subseteq I_j$ for some $j \in \{1, ..., n\}$ where I_i is an ideal of R and R is a um-ring if for each R-module M with $M = \cup_{i=1}^{n} N_i$, then $M = N_j$ for some $j \in \{1, ..., n\}$ where N_i is a submodule of M.

Marcelo and Masque ([18], p. 273) proved that a proper submodule N of an R-module M is a prime submodule of M if and only if the natural homomorphism $\theta_r : \frac{M}{N} \to \frac{M}{N}$ by $\theta_r(m+N) = rm+N$ is zero or one-to-one. Furthermore, let F be a flat R-module and N be a prime submodule of M. Azizi ([9], Lemma 3.2) showed that if $F \otimes N$ is a proper submodule of $F \otimes M$, then $F \otimes N$ is a prime submodule of $F \otimes M$. Also, for every flat R-module F we have $F \otimes (P : r) = (F \otimes P : r)$ where P is a submodule of M. Ebrahimpour and Mirzaee proved that if F is a flat R-module and N is a weakly semiprime submodule of M with $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly semiprime submodule of $F \otimes M$ (see [12],Theorem 2.22)). Now, let R be a um-ring, M be an R-module, F be a flat R-module and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Assume that N is a proper submodule of M such that $\varphi(F \otimes N) = F \otimes \varphi(N)$. If N is a φ-classical prime submodule of M with $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a φ-classical prime submodule of $F \otimes M$ (see ([20], Theorem 2.18)). We state Theorem 2.18 in [20] for $(n-1,n)$-φ-classical prime submodules. Firstly, we state two propositions similar to Theorem 2.11 and Theorem 2.14 in [20].

Proposition 2.19. *Let M be an R-module, N be a proper submodule of M and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ a function.The following conditions are equivalent:*
(i) N is an $(n-1,n)$-φ-classical prime submodule of M;
(ii) For every $r_1, \ldots, r_{n-1} \in R$, $(N :_M r_1 \ldots r_{n-1}) = (\varphi(N) :_M r_1 \ldots r_{n-1})$ $\cup (N :_M r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1})$, for some $i \in \{1, ..., n-1\}$.

Proof. $(i) \Rightarrow (ii)$ Let $m \in (N :_M r_1 \ldots r_{n-1})$, so $r_1 \ldots r_{n-1} m \in N$. If $r_1 \ldots r_{n-1} m \in \varphi(N)$, then $m \in (\varphi(N) :_M r_1 \ldots r_{n-1})$. Now, suppose that $r_1 \ldots r_{n-1} m \notin \varphi(N)$ and hence $r_1 \ldots r_{n-1} m \in N \setminus \varphi(N)$. Since N is an $(n-1,n)$-φ-classical prime submodule of M, so $r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1} m \in N$. Thus $m \in (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$, for some $i \in \{1, ..., n-1\}$. Conversely, if $m \in (\varphi(N) :_M r_1 \ldots r_{n-1}) \cup (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$ for some $i \in \{1, ..., n-1\}$. Since $\varphi(N) \subseteq N$, therefore $m \in (N :_M r_1 \ldots r_{n-1})$.
$(ii) \Rightarrow (i)$ Let $r_1 \ldots r_{n-1} m \in N \setminus \varphi(N)$ whenever $r_1, \ldots, r_{n-1} \in R$ and $m \in M$. Then $m \in (N :_M r_1 \ldots r_{n-1})$ and $m \notin (\varphi(N) :_M r_1 \ldots r_{n-1})$. By our assumption, we have $m \in (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$ for some $i \in \{1, ..., n-1\}$. Accordingly, N is an $(n-1,n)$-φ-classical prime submodule of M. □

Proposition 2.20. *Let M be an R-module, R be a um-ring and N be a proper submodule of M. Asuume that $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*
(i) N is an $(n-1,n)$-φ-classical prime submodule of M;
(ii) For every $r_1, \ldots, r_{n-1} \in R$, $(N :_M r_1 \ldots r_{n-1}) = (\varphi(N) :_M r_1 \ldots r_{n-1})$ or $(N :_M r_1 \ldots r_{n-1}) = (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$ for some $i \in \{1, ..., n-1\}$.

Proof. By Proposition 2.19 and Definition 2.18, the proof is clear. □

Theorem 2.21. *Suppose that M be an R-module, R be a um-ring and $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let N be a proper submodule of M and F be a flat R-module with $\varphi(F \otimes N) = F \otimes \varphi(N)$. If N is an $(n-1,n)$-φ-classical prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is an $(n-1,n)$-φ-classical prime submodule of $F \otimes M$.*

Proof. Let $r_1, \ldots, r_{n-1} \in R$, by Proposition 2.20, we have $(N :_M r_1 \ldots r_{n-1}) = (\varphi(N) :_M r_1 \ldots r_{n-1})$ or $(N :_M r_1 \ldots r_{n-1}) = (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$. If $(N :_M r_1 \ldots r_{n-1}) = (\varphi(N) :_M r_1 \ldots r_{n-1})$ by ([8], Lemma 3.2), $(F \otimes N :_{F \otimes M} r_1 \ldots r_{n-1}) = F \otimes (N :_M r_1 \ldots r_{n-1}) = (F \otimes \varphi(N) :_{F \otimes M} r_1 \ldots r_{n-1}) = (\varphi(F \otimes N) :_{F \otimes M} r_1 \ldots r_{n-1})$. Now, assume that $(N :_M r_1 \ldots r_{n-1}) = (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$, by ([8], Lemma 3.2), $(F \otimes N :_{F \otimes M} r_1 \ldots r_{n-1}) = F \otimes (N :_M r_1 \ldots r_{n-1}) = F \otimes (N :_M r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1}) = (F \otimes N :_{F \otimes M} r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1})$. Accordingly, by Proposition 2.20, we conclude that $F \otimes N$ is an $(n-1,n)$-φ-classical prime submodule of $F \otimes M$. □

Definition 2.22. A proper submodule N of M is called finitely compactly packed if for each family $\{P_\alpha\}_{\alpha \in \Lambda}$ of prime submodules of M with $N \subseteq \cup_{\alpha \in \Lambda} P_\alpha$, there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $N \subseteq \cup_{i=1}^{n} P_{\alpha_i}$. If $N \subseteq P_\beta$ for some $\beta \in \Lambda$, then N is called compactly packed . A module

M is said to be finitely compactly packed (compactly packed), if every proper submodule N of M is finitely compactly packed (compactly packed) submodule.

We will call a proper submodule N of M as $(n-1,n)$-φ-*classical prime finitely compactly packed* (or abriviated by $(n-1,n)$-φ-$CFCP$ submodule) if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of $(n-1,n)$-φ-classical prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$, there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $N \subseteq \cup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called $(n-1,n)$-φ-*classical prime compactly packed* (or abriviated by $(n-1,n)$-φ- CCP submodule) . A module M is said to be $(n-1,n)$-φ-$CFCP$ ($(n-1,n)$-φ-CCP) if every proper submodule is an $(n-1,n)$-φ-$CFCP$ ($(n-1,n)$-φ-CCP) submodule of M.

For more details concerning finitely compactly packed (compactly packed) submodule of a module refer to [1], [7] and [21].

Proposition 2.23. *Let M be an R-module, φ_1 and $\varphi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions with $\varphi_1 \leq \varphi_2$ (i.e., for every of submodule N, $\varphi_1(N) \subseteq \varphi_2(N)$). If M is an $(n-1,n)$-φ_2-CCP ($(n-1,n)$-φ_2-$CFCP$) module, then M is an $(n-1,n)$-φ_1-CCP ($(n-1,n)$-φ_1-$CFCP$) module.*

Proof. It is clear directly from the definition that every $(n-1,n)$-φ_1-classical prime submodule of M is an $(n-1,n)$-φ_2-classical prime submodule of M. Now, assume that N be a proper submodule of M with $N \subseteq \cup_{\alpha\in\Lambda} P_\alpha$ where P_α is an $(n-1,n)$- φ_1-classical prime submodule of M. Since P_α is an $(n-1,n)$- φ_2-classical prime submodule of M for each $\alpha \in \Lambda$, so $N \subseteq P_\beta$ for some $\beta \in \Lambda$, because M is an $(n-1,n)$-φ_2-CCP module. Since P_β is an $(n-1,n)$-φ_1-classical prime submodule of M and hence N is an $(n-1,n)$-φ_1-CCP submodule of M. Thus M is an $(n-1,n)$-φ_1-CCP module. Similarly, we can prove that if M is an $(n-1,n)$-φ_2-$CFCP$ module, then M is an $(n-1,n)$-φ_1-$CFCP$ module. □

Proposition 2.24. *Every $(n,n+1)$-φ-CCP ($CFCP$) module is an $(n-1,n)$-φ-CCP ($CFCP$) module.*

Proof. The proof is straightforward. □

Theorem 2.25. *Let $f : M \to M'$ be an epimorphism R-module, $\varphi : S(M) \to S(M) \cup \{\emptyset\}$ and $\varphi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*

(i) If M is an $(n-1,n)$-φ-$CFCP$ (CCP) module such that for every $(n-1,n)$-φ'- classical prime submodule L of M' with $f^{-1}(\varphi'(L)) \subseteq \varphi(f^{-1}(L))$, then M' is an $(n-1,n)$-φ'-$CFCP$ (CCP) module.

(ii) If M' is an $(n-1,n)$-φ'-$CFCP$ (CCP) module such that every $(n-1,n)$-φ-classical prime

submodule N of M with $kerf \subseteq N$ and $f(\varphi(N)) \subseteq \varphi'(f(N))$, then M is an $(n-1,n)$-φ-$CFCP$ (CCP) module.

Proof. (i) Assume that P' be a proper submodule of M' with $P' \subseteq \bigcup_{\alpha\in\Lambda} L'_\alpha$ where L'_α is an $(n-1,n)$-φ'-classical prime submodule of M' for each $\alpha \in \Lambda$. We get $f^{-1}(P') \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(L'_\alpha)$. Since L'_α is an $(n-1,n)$-φ'-classical prime submodule of M' for each $\alpha \in \Lambda$ and $f^{-1}(\varphi'(L'_\alpha) \subseteq \varphi(f^{-1}(L'_\alpha))$, by Proposition 2.11, we have $f^{-1}(L'_\alpha)$ is an $(n-1,n)$-φ-classical prime submodule of M for each $\alpha \in \Lambda$. Since M is an $(n-1,n)$-φ-$CFCP$ module, therefore there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $f^{-1}(P') \subseteq \bigcup_{i=1}^n f^{-1}(L'_{\alpha_i})$ and hence $f^{-1}(P') \subseteq f^{-1}(\bigcup_{i=1}^n L'_{\alpha_i})$. Since f is an epimorphism R-module, so $P' \subseteq \bigcup_{i=1}^n L'_{\alpha_i}$. It follows that P' is an $(n-1,n)$-φ'-$CFCP$ submodule of M'. Therefore M' is an $(n-1,n)$-φ'-$CFCP$ module. Simiarly, we can show that P' is an $(n-1,n)$-φ'-CCP submodule of M'. Thus M' is an $(n-1,n)$-φ'-CCP module.
(ii) Let P be a proper submodule of M such that $P \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is an $(n-1,n)$-φ-classical prime submodule of M for each $\alpha \in \Lambda$. We have $f(P) \subseteq f(\bigcup_{\alpha\in\Lambda} N_\alpha) = \bigcup_{\alpha\in\Lambda} f(N_\alpha)$. Since N_α is an $(n-1,n)$-φ-classical prime submodule of M, $f(\varphi(N_\alpha)) \subseteq \varphi'(f(N_\alpha))$ and $kerf \subseteq N_\alpha$ for each $\alpha \in \Lambda$, by Proposition 2.11, we get $f(N_\alpha)$ is an $(n-1,n)$-φ'-classical prime submodule of M'. Since M' is an $(n-1,n)$-φ'-$CFCP$ module, thus there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $f(P) \subseteq \bigcup_{i=1}^n f(N_{\alpha_i})$. Now, we show that $P \subseteq \bigcup_{i=1}^n N_{\alpha_i}$. Let $p \in P$ and hence $f(p) \in f(\bigcup_{i=1}^n N_{\alpha_i})$. We have $f(p) = f(q)$ for some $q \in \bigcup_{i=1}^n N_{\alpha_i}$. Therefore $p - q \in kerf \subseteq N_{\alpha_j}$ and $q \in N_{\alpha_j}$ for some $\alpha_j \in \{\alpha_1, ..., \alpha_n\}$. Thus $p \in N_{\alpha_j}$ and hence $p \in \bigcup_{i=1}^n N_{\alpha_i}$. This shows that P is an $(n-1,n)$-φ-$CFCP$ submodule of M and so M is an $(n-1,n)$-φ-$CFCP$ module. Likewise, we can prove that P is an $(n-1,n)$-φ-CCP submodule of M and hence M is an $(n-1,n)$-φ-CCP module. □

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References

[1] Z. Al-Ani, *Compactly packed modules and comprimely packed modules*, M.sc. Theses, College of Science, University of Baghdad (1998).

[2] R. Ameri, *On the prime submodules of multiplication modules*, Inter. J. Math. Math. Sci. **27** (2003),1715-1724.

[3] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings* , Comm. Algebra. **39** (2011), 1646-1672.

[4] D. F. Anderson and A. Badawi, *On (m,n)-closed ideals of commutative rings*, J. Algebra Appl. 16 (1) (2017), 1750013, 21pp.

[5] D. D. Anderson and M. Batanieh, *Generalizations of prime ideals*, Comm. Algebra. **36** (2008), 686-696.

[6] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (2003), 831-840.

[7] A. Ashour, *Primary Finitely Compactly Packed Modules and S-Avoidance Theorem for Modules*, Turkish J. Math. **32** (2008), 315-324.

[8] A. Azizi, *Prime submodules and flat modules*, Acta Math. Sin. (Engl. Ser.) **23** (1) (2007). 147-152.

[9] A. Azizi, *Weakly prime submodules and prime submodules*, Glasg. Math. J. **48** (2) (2006), 343-346.

[10] A. Y. Darani and F. Soheilnia, *On n-absorbing submdules*, Math. Commun. **17** (2012), 547-557.

[11] J. Dauns, *Prime submodules* , J. Reine Angew. Math. **298** (1978) 156-181.

[12] M. Ebrahimpour and F. Mirzaee, *On φ-Semiprime Submodules*, J. Korean Math. Soc. **54**, No. 4 (2017) 1099-1108.

[13] M. Ebrahimpour and R. Nekooei, *On generalizations of prime submodules*, Bull. Iranian Math. Soc. **39** (5)(2013), 919-939.

[14] Z. A. El-Bast and P.F.Smith, *Multiplication modules*, Comm. Algebra. **16** (1988), 766-779.

[15] A. K. Jabbar, *A generalization of prime and weakly prime submodules*, Pure Math. Sci. **2** (2013), 1-11.

[16] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli **33** (1984), 61-69.

[17] C. P. Lu, *Spectra of modules*, Comm. Algebra. **23** (1995), 3741-3752.

[18] Marcelo, A., Masque, M., *Prime submodule, the descent invariant, and module of finite length*, J. Algebra. **189** (1997), 273-293.

[19] R. L. McCasland and M.E. Moore, *Prime submodules*, Comm. Algebra. **20** (1992), 1803-1817.

[20] H. Mostafanasab, E.S. Sevim, S. Babaei and U.Tekir, *φ-Classical prime submodules*, arXiv:1507.08981v1 [math.AC] 30 Jul 2015.

[21] J. V. Pakala and T.S. Shores, *On Compactly Packed Rings*, Pacific J. Math. Vol. **97**, No. 1 (1981), 197-201.

[22] P. Quartararo and H.S. Butts, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc. **52** (1975), 91-96.

[23] P.F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50** (1988), 223-235.

[24] N. Zamani, *φ-prime submodules*, Glasg. Math. J. **52**(2)(2010), 253-259.

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