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ON THE EIGENVALUES OF CAYLEY GRAPHS ON GENERALIZED DIHEDRAL GROUPS

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Abstract. Let Γ be a graph with adjacency eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then the energy of Γ, a concept defined in 1978 by Gutman, is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. Also the Estrada index of Γ, which is defined in 2000 by Ernesto Estrada, is defined as $EE(\Gamma) = \sum_{i=1}^{n} e^{\lambda_i}$. In this paper, we compute the eigenvalues, energy and Estrada index of Cayley graphs on generalized dihedral groups. As an application, we compute these items for honeycomb toroidal graphs and Cayley graphs on dihedral groups.

1. Introduction

In this paper, graphs are finite, undirected and simple. Also groups are finite. Let Γ be a finite simple graph with n vertices. The *adjacency matrix* A of Γ is an $n \times n$ matrix where its rows and columns are indexed by vertices of Γ and its (u, v)-entry is 1 whenever the vertices u and v are adjacent and 0 otherwise. The characteristic polynomial of A, $p_A(\lambda)$ is defined as $\det(\lambda I - A)$. The roots of $p_A(\lambda)$ are called *eigenvalues* of Γ. Since Γ is undirected, its adjacency matrix is symmetric and so its eigenvalues are real. Spectral graph theory has

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many applications such as the study of the stability of molecules [8]. The energy of a graph, a concept of chemistry which is associated with the eigenvalues of graph, is defined in 1978 by Gutman as the sum of the absolute values of the eigenvalues of the graph [8]. Another quantity, related to the eigenvalues of a graph, which has many applications in chemistry and computer science, the Estrada index of a graph, is defined in 2000 by Ernesto Estrada [6].

Let G be a finite group and $T = T^{-1} \subseteq G \setminus \{1\}$. The *Cayley graph on G with respect to T*, $\mathrm{Cay}(G,T)$, is a graph with vertex set G and edge set $\{\{g, tg\} \mid g \in G, t \in T\}$. By a theorem of Sabidussi [13], a graph Γ is a Cayley graph on G if its automorphism group contains a regular subgroup, on the vertex of Γ, isomorphic to G. There are many famous graphs which are Cayley graphs, such as complete graphs and cycles.

In 1975, Lovász [11] proved that the computation of eigenvalues of any vertex-transitive graph can be reduced to that of some Cayley graph. Babai [4] proposed an explicit formula for the spectrum of a Cayley graph on G in terms of irreducible characters of G. Then, Diaconis and Shahshahani [5] presented the adjacency matrix of a Cayley graph on a group G in terms of irreducible representations of G and determined its eigenvalues, see also [2, Corollary 7]. Although determining the eigenvalues of Cayley graphs reduces to finding irreducible representations or characters, determining irreducible representations and characters of finite groups is not easy in general. But the irreducible representations and characters of finite abelian groups are fully determined. Semi-Cayley graphs which are a generalization of Cayley graphs defined by Resmini and Jungnickel [12], see also [3]. A graph Γ is called a *semi-Cayley graph over* G if its automorphism group contains a semiregular subgroup isomorphic to G with two orbits. Let Γ be a semi-Cayley graph over a group G. Then there exist subsets R, L and S of a G such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of G such that $\Gamma \cong \mathrm{SC}(G; R, L, S)$, where $\mathrm{SC}(G; R, L, S)$ is an undirected graph with vertex set $G \times \{1, 2\}$ and its edge set consists of three sets:

$$\{\{(x,1),(y,1)\} \mid yx^{-1} \in R\} \quad \text{(right edges)},$$
$$\{\{(x,2),(y,2)\} \mid yx^{-1} \in L\} \quad \text{(left edges)},$$
$$\{\{(x,1),(y,2)\} \mid yx^{-1} \in S\} \quad \text{(spoke edges)}.$$

A generalized dihedral group is a semidirect product of an abelian group with a subgroup of order 2. Hence it has an abelian subgroup of index 2. It has been proved in [2, Lemma 8] that every Cayley graph on a group G having a subgroup H of index 2 is a *semi-Cayley graph* over H. Eigenvalues of semi-Cayley graphs are determined in [2, Theorem 6]. By the motivation of [7], we use [3, Theorem 6] to determine the eigenvalues of Cayley graphs on generalized dihedral groups, their energy and Estrada index which are the aim of paper. For the group theoretic and representation theoretic terminology not defined here, we refer the reader to [10] and [9], respectively.

2. Main result

Let us recall some representation theory of finite groups which will be used in this paper, for more details see [9]. Let G be a finite group. A (complex) *representation of G of degree n* is a group homomorphism $\rho : G \to \mathrm{GL}(n, \mathbb{C})$. A (complex) *character* of G afforded by ρ is the function given by $\chi : G \to \mathbb{C}$, where $\chi(g)$ is equal to the trace of $\rho(g)$. The representation ρ is called *irreducible* if every $n \times n$ matrix A which satisfies $\rho(g)A = A\rho(g)$ for all $g \in G$ has the form $A = \lambda I_n$, where $\lambda \in \mathbb{C}$ and I_n is the identity matrix of rank n. The character χ is called irreducible if ρ is irreducible. Two representations ρ and σ of same degree n are called *equivalent* if there exists an invertible $n \times n$ matrix T such that for all $g \in G$, $T\sigma(g) = \rho(g)T$. We denote the set of all inequivalent irreducible representations and inequivalent irreducible characters of G by $\mathrm{irr}(G)$ and $\mathrm{Irr}(G)$, respectively. The size of $\mathrm{Irr}(G)$ is equal to the number of conjugacy classes of G [9, Corollary 2.7]. In general, it is not easy to find the set of inequivalent irreducible representations and characters of a given group. If G is abelian then all of its irreducible representations are of degree 1 and so $\mathrm{irr}(G) = \mathrm{Irr}(G)$.

In this paper, we need irreducible characters of finite abelian groups. Irreducible characters of these groups are well-known. Here we recall it. Let G be a finite abelian group. Then [10, Theorem 2.1.3] implies that $G = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \times \langle g_n \rangle$. Let g_k has order n_k. Then irreducible characters of $\langle g_k \rangle$ are $\chi_j : \langle g_k \rangle \to \mathbb{C}$, where $\chi_j(g_k^l) = e^{\frac{2\pi i j l}{n_k}}$ and $j, l = 0, \ldots, n_k - 1$. Hence by [9, Theorem 4.2.1], irreducible characters of G are

$$\chi_{j_1,\ldots,j_n} : G \to \mathbb{C},\ \chi_{j_1,\ldots,j_n}(g_1^{l_1} \ldots g_n^{l_n}) = e^{2\pi i \sum_{r=1}^{n} \frac{l_r j_r}{n_r}},\ 0 \leq j_r \leq n_r - 1, 1 \leq r \leq n.$$

Let f be a representation or a character of G of degree n and X be a subset of G. Then we denote $\sum_{x \in X} f(x)$ by $f(X)$. If $X = \emptyset$, we put $f(X) = O$ or 0, whenever f is a representation or a character, respectively, where O is the $n \times n$ zero matrix.

First let us start with an easy lemma which says that any Cayley graph over a group having a subgroup of index 2 is a semi-Cayley graph over that subgroup.

Lemma 2.1. *Let $\Gamma = \mathrm{Cay}(G, T)$ be a Cayley graph on a group G having a subgroup H of index 2. Let $\{1, a\}$ be a left transversal to H in G. Then $\Gamma \cong \mathrm{SC}(H; R, aRa^{-1}, S)$, where $R = H \cap T$ and $S = H \cap aT$.*

Proof. By [2, Lemma 8], $\Gamma \cong \mathrm{SC}(H; R, L, S)$, where $R = H \cap T$, $L = H \cap aTa^{-1}$ and $S = H \cap aT$. Since H is of index 2, it is a normal subgroup of G. Hence $aHa^{-1} = H$. Thus $L = aHa^{-1} \cap aTa^{-1} = a(H \cap T)a^{-1} = aRa^{-1}$, which completes the proof. ∎

In the following theorem we determine the eigenvalues of Cayley graphs over finite groups having a subgroup of index 2.

Theorem 2.2. *Let G be a group having a subgroup H of index 2 with left transversal $\{1, a\}$ in G, $\mathrm{irr}(H) = \{\rho_1, \ldots, \rho_m\}$ and $\Gamma = \mathrm{Cay}(G, T)$. Then the set of eigenvalues of Γ is the union of the set of eigenvalues of A_l, $l = 1, \ldots, m$, where*

$$A_l = \begin{bmatrix} \rho_l(R) & \rho_l(S) \\ \rho_l(S^{-1}) & \rho_l(aRa^{-1}) \end{bmatrix},$$

$R = H \cap T$ and $S = H \cap Ta^{-1}$. Furthermore, if λ is an eigenvalue of A_l with multiplicity of $m_l(\lambda)$ then λ is an eigenvalue of Γ with multiplicity $\sum_{l=1}^{m} d_l m_l(\lambda)$, where d_l is the degree of ρ_l.

Proof. By Lemma 2.1, $\Gamma \cong \mathrm{SC}(H; R, aRa^{-1}, S)$, where $R = H \cap T$ and $S = H \cap aT$. Now, putting $n = 2$ in [2, Theorem 6], we can get the result. ■

In the following result, we assume that H is abelian in Theorem 2.2.

Corollary 2.3. *Let G be a finite group having an abelian subgroup H of index 2, $\{1, a\}$ be a left transversal of H in G and $\mathrm{Irr}(H) = \{\chi_1, \ldots, \chi_m\}$ and $\Gamma = \mathrm{Cay}(G, T)$. Then eigenvalues of Γ are*

$$\chi_l(R) + |\chi_l(S)|, \ \chi_l(R) - |\chi_l(S)|$$

where $l = 1, \ldots, m$, $R = H \cap T$ and $S = H \cap Ta^{-1}$.

Proof. First note that since H is abelian, $\mathrm{Irr}(H) = \mathrm{irr}(H)$. Furthermore, for each $\chi \in \mathrm{Irr}(H)$ we have $\chi(aRa^{-1}) = \sum_{r \in R} \chi(ara^{-1}) = \sum_{r \in R} \chi(r) = \chi(R)$ and $\chi(S^{-1}) = \sum_{s \in S} \chi(s^{-1}) = \sum_{s \in S} \overline{\chi(s)} = \overline{\chi(S)}$. Now, by Theorem 2.2, the set of eigenvalues of Γ is the union of the set of eigenvalues of matrices

$$A_l = \begin{bmatrix} \chi_l(R) & \chi_l(S) \\ \overline{\chi(S)} & \chi_l(R) \end{bmatrix},$$

where $l = 1, \ldots, m$. On the other hand, A_l has eigenvalues $\chi_l(R) + |\chi_l(S)|$ and $\chi_l(R) - |\chi_l(S)|$, which completes the proof. ■

The following example is a direct consequence of Corollary 2.3 and the fact that irreducible characters of $H = \langle x \rangle \cong \mathbb{Z}_n$ are $\chi_0, \ldots, \chi_{n-1}$, where $\chi_k(x^l) = e^{\frac{2\pi i l k}{n}}$.

Example 2.4. Let $G = \langle x, y \mid x^n = y^2 = (xy)^2 = 1 \rangle$ be the dihedral group of order $2n$, $T = T^{-1} \subseteq G \setminus \{1\}$ and $\Gamma = \mathrm{Cay}(G, T)$. Then one of the following happens:

(1) $T = \{x^{i_1}y, \ldots, x^{i_k}y\}$. In this case eigenvalues of Γ are $\pm k$ and $\pm|\sum_{r=1}^{k} e^{\frac{2\pi i l i_r}{n}}|$, where $l = 1, \ldots, n-1$.

(2) $T=\{x^{i_1},\ldots,x^{i_k}\}$. In this case eigenvalues of Γ are $\pm k$ and $\sum_{r=1}^{k} e^{\frac{2\pi i l i_r}{n}}$ with multiplicity 2, where $l=1,\ldots,n-1$.

(3) $T=\{x^{i_1},\ldots,x^{i_r},x^{j_1}y,\ldots,x^{j_s}y\}$. In this case eigenvalues of Γ are $r+s$, $r-s$ and $\sum_{t=1}^{r} e^{\frac{2\pi i l i_t}{n}} \pm |\sum_{t=1}^{s} e^{\frac{2\pi i l j_t}{n}}|$, where $l=1,\ldots,n-1$.

Corollary 2.5. *Keeping the notations of Corollary 2.3, the Estrada index of* Γ, $EE(\Gamma)$, *is equal to* $2\sum_{l=1}^{m} e^{\chi_l(R)}\cosh(|\chi_l(S)|)$, *where* $R=H\cap T$ *and* $S=H\cap Ta^{-1}$. *Furthermore,* $\mathcal{E}(\Gamma)\le |G||T|-2\min\{|R|,|S|\}$.

Proof. By Corollary 2.3, eigenvalues of Γ are $\chi_l(R)+|\chi_l(S)|$, $\chi_l(R)-|\chi_l(S)|$ where $l=1,\ldots,m$, $R=H\cap T$ and $S=H\cap Ta^{-1}$. For real numbers x,y we have $e^{x-y}+e^{x+y}=e^x(e^y+e^{-y})=2e^x\cosh(y)$. Also $EE(\Gamma)=\sum_{\lambda\in \mathrm{Spec}(\Gamma)} e^{\lambda}$. To complete the proof, it is enough to replace the eigenvalues.

Now we find the upper bond of the energy of Γ. Let χ_1 be the trivial character of H. Then $\chi_1(R)=|R|$, $\chi_1(S)=|S|$ and we have

$$\begin{aligned}
\mathcal{E}(\Gamma) &= \sum_{\lambda\in \mathrm{Spec}(\Gamma)} |\lambda| \\
&= \sum_{l=1}^{m}\Big(|\chi_l(R)+|\chi_l(S)||+|\chi_l(R)-|\chi_l(S)||\Big) \\
&= 2\max\{|R|,|S|\}+\sum_{l=2}^{m}\Big(|\chi_l(R)+|\chi_l(S)||+|\chi_l(R)-|\chi_l(S)||\Big) \\
&\le 2\max\{|R|,|S|\}+2\sum_{l=2}^{m}\Big(|\chi_l(R)|+|\chi_l(S)|\Big) \text{ (by the triangular inequality).}
\end{aligned}$$

On the other hand, for $l\ge 2$ we have $|\chi_l(R)|=|\sum_{r\in R}\chi_l(r)|\le \sum_{r\in R}|\chi_l(r)|$ and, by [9, Lemma 2.15(c)], for all $r\in R$, $|\chi_l(r)|\le 1$. Hence $|\chi_l(R)|\le |R|$. Similarly, $|\chi_l(S)|\le |S|$. Hence $\mathcal{E}(\Gamma)\le 2\max\{|R|,|S|\}+2(m-1)(|R|+|S|)$. On the other hand, $2m=2|H|=|G|$ and $|R|+|S|=|H\cap T|+|H\cap Ta^{-1}|=|H\cap T|+|(Ha\cap T)a^{-1}|=|H\cap T|+|Ha\cap T|=|T|$. Hence $\mathcal{E}(\Gamma)\le 2\max\{|R|,|S|\}+|G||T|-2(|R|+|S|)=|G||T|-2\min\{|R|,|S|\}$. This completes the proof. ■

Given an abelian group H, the *generalized dihedral group* $D_H=\langle H,a\rangle$ is a group generated by H and an element a such that $a\notin H$, $a^2=1$ and $aha=h^{-1}$ for all $h\in H$. It is easy to see that H is a normal subgroup of G of index 2. So we have the following result.

Corollary 2.6. *Let* $H=\langle x_1\rangle\times\ldots\times\langle x_m\rangle$ *be a finite abelian group,* x_i *has order* n_i, $G=\langle H,a\rangle$ *be a generalized dihedral group, and suppose that* $\Gamma=\mathrm{Cay}(G,T)$, *where* $T=\{a,h_1a,h_2a\}$ *for* *some elements* $h_1 = x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}$ *and* $h_2 = x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}$ *of* H*. Then eigenvalues of* Γ *are*

$$\pm\sqrt{3+2\Big(\cos(2\pi\sum_{k=1}^{m}\frac{(i_k-j_k)l_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{i_kl_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{j_kl_k}{n_k})\Big)},$$

where for each k*,* $l_k \in \{0,\ldots,n_k-1\}$*. In particular,* $EE(\Gamma)$ *is equal to*

$$2\sum_{k=1}^{m}\sum_{l_k=0}^{n_k-1}\cosh\sqrt{3+2\Big(\cos(2\pi\sum_{k=1}^{m}\frac{(i_k-j_k)l_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{i_kl_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{j_kl_k}{n_k})\Big)},$$

and the energy of Γ *is equal to*

$$2\sum_{k=1}^{m}\sum_{l_k=0}^{n_k-1}\sqrt{3+2\Big(\cos(2\pi\sum_{k=1}^{m}\frac{(i_k-j_k)l_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{i_kl_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{j_kl_k}{n_k})\Big)}.$$

Proof. First note that since $a \notin H$, we have $H \cap T = \emptyset$. Also $H \cap Ta^{-1} = \{1, h_1, h_2\}$. Let χ be a fixed irreducible character of H. Then there exists integers $l_1, \ldots, l_m$, where $0 \le l_k \le n_k - 1$, such that $\chi(h_1) = e^{2\pi i \sum_{k=1}^{m} \frac{i_k l_k}{n_k}}$ and $\chi(h_2) = e^{2\pi i \sum_{k=1}^{m} \frac{j_k l_k}{n_k}}$. This implies that

$$|1+\chi(h_1)+\chi(h_2)| = \sqrt{3+2\Big(\cos(\theta_1-\theta_2)+\cos(\theta_1)+\cos(\theta_2)\Big)},$$

where $\theta_1 = 2\pi\sum_{k=1}^{m}\frac{i_kl_k}{n_k}$ and $\theta_2 = 2\pi\sum_{k=1}^{m}\frac{j_kl_k}{n_k}$. Thus, by Corollary 2.3, eigenvalues of Γ are

$$\pm\sqrt{3+2\Big(\cos(2\pi\sum_{k=1}^{m}\frac{(i_k-j_k)l_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{i_kl_k}{n_k})+\cos(2\pi\sum_{k=1}^{m}\frac{j_kl_k}{n_k})\Big)},$$

where $0 \le l_k \le n_k$. The second part is a direct consequence of the first part. ■

Now we are ready to compute the eigenvalues, energy and Estrada index of honeycomb toroidal graphs as an application of Corollary 2.6. The *honeycomb toroidal graph* $HTG(m, 2n, s)$, where m, n and s are positive integers, $n > 1$ and $m + s$ is even, is a trivalent graph which is called *generalized honeycomb tori* by some authors, has been studied in several contexts. These graphs have been suggested as an attractive architecture for interconnection networks in parallel and distributed applications, and have been studied by various authors [1].

Corollary 2.7. *We have exact formulas for eigenvalues, Estrada index and energy of any honeycomb toroidal graph.*

Proof. Let $\Gamma = HTG(m, 2n, s)$ be a honeycomb toroidal graph. By [1, Theorem 3.4], Γ is a Cayley graph on generalized dihedral group $D_H = \langle H, \tau\rangle$, where $H = \langle x, y\rangle$, $x^n = \tau^2 = 1$, $xy = yx$, $y^m = x^{(m+s)/2}$, $\tau x\tau = x^{-1}$ and $\tau y\tau = y^{-1}$, with respect to $T = \{\tau, \tau x, \tau y\}$. Since H is abelian, there exist elements $x_1, \ldots, x_m$ of H such that x_i has order n_i and $H = \langle x_1\rangle \times \ldots \times \langle x_m\rangle$. Let $x = x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}$ and $y = x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}$. Then one can exactly determine eigenvalues, Estrada index and energy of Γ using Corollary 2.6. ■

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References

[1] B. Alspach and M. Dean, *Honeycomb toroidal graphs are Cayley graphs*, Information Processing Letters, Vol. 109 (2009), pp. 705–708.

[2] M. Arezoomand and B. Taeri, *On the characteristic polynomial of n-Cayley digraphs*, Electron. J. Combin. Vol. 20 No.3 (2013), P57, pp. 1–14.

[3] M. Arezoomand and B. Taeri, *Normality of 2-Cayley digraphs*, Discrete Math., Vol. 338 (2015), pp. 41–47.

[4] L. Babai, *Spectra of Cayley graphs*, J. Combin. Theory Ser. B, Vol. 27 (1979), pp. 180–189.

[5] P. Diaconis and M. Shahshahani, *Generating a random permutation with random transpositions*, Z. Wahr. Verw. Gebiete, Vol. 57 (1981), pp. 159–179.

[6] E. Estrada, *Characterization of 3D molecular structure*, Chem. Phys. Lett. Vol. 319 (2000), pp. 713–718.

[7] M. Ghorbani, *On the energy and Estrada index of Cayley graphs*, Discrete Mathematics, Algorithms and Applications, Vol. 7 No. 1 (2015) , 155005.

[8] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forschungsz. Graz Vol. 103 (1978), pp. 1–22.

[9] M. Isaacs, *Character theory of finite groups*, Academic Press (1976).

[10] H. Kurzwil and B. Stellmacher, *The theory of finite groups, An introduction*, Springer-Verlag (2004).

[11] L. Lovász, *Spectra of graphs with transitive groups*, Period. Math. Hungar. Vol. 6 (1975), pp. 191–196.

[12] M. J. de Resmini, and D. Jungnickel, *Strongly regular semi-Cayley graphs*, J. Algebraic Combin., Vol. 1 (1992), pp. 217–228.

[13] G. Sabidussi. *Vertex-transitive graphs* Monatsh. Math., Vol. 68 (1964), pp. 426–438.

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