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THE AUTOMORPHISM GROUP OF THE REDUCED COMPLETE-EMPTY $X-$JOIN OF GRAPHS

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Abstract. Suppose X is a simple graph. The $X-$join Γ of a set of complete or empty graphs $\{X_x\}_{x\in V(X)}$ is a simple graph with the following vertex and edge sets:

$$V(\Gamma) = \{(x, y) \mid x \in V(X) \ \&\ y \in V(X_x)\},$$
$$E(\Gamma) = \{(x, y)(x', y') \mid xx' \in E(X) \ or\ else\ x = x' \ \&\ yy' \in E(X_x)\}.$$

The $X-$join graph Γ is said to be reduced if $x, y \in V(X)$, $x \neq y$ and $N_X(x)\setminus\{y\} = N_X(y)\setminus\{x\}$ imply that (i) if $xy \notin E(X)$ then the graphs X_x or X_y are non-empty; (ii) if $xy \in E(X)$ then X_x or X_y are not complete graphs. The aim of this paper is to explore how the graph theoretical properties of $X-$join of graphs effect on its automorphism group. Among other results we compute the automorphism group of reduced complete-empty $X-$join of graphs.

1. Introduction

Throughout this paper all graphs are assumed to be simple and undirected. Our notations are standard and taken mainly from [8, 9]. Suppose X is such a graph. Sabidussi [2, p. 396],

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has defined the $X-$join of a set of graphs $\{X_x\}_{x\in V(X)}$ as the graph Γ obtained by replacing each vertex $x \in V(X)$ by the graph X_x and inserting either all or none of the possible edges between vertices of X_x and X_y depending on whether or not x and y are joined by an edge in X. In fact, the vertex and edge sets of Γ are defined as below:

$$\begin{aligned} V(\Gamma) &= \{(x,y) \mid x \in V(X) \ \&\ y \in V(X_x)\}, \\ E(\Gamma) &= \{(x,y)(x',y') \mid xx' \in E(X) \ \textit{or else}\ x = x' \ \&\ yy' \in E(X_x)\}. \end{aligned}$$

In this paper, X is assumed to be connected and the $X-$join of complete or empty graphs X_x, $x \in V(X)$, is denoted by $(\biguplus_{x\in V(X)} X_x)_X$. It is clear that when $X = K_2$, the $X-$join of graphs X_1 and X_2 is the ordinary join and if $X = P_n$, $n \geq 2$, then the $X-$join of graphs $X_1, \cdots, X_{n+1}$ is the sequential join of these graphs.

Suppose Δ is a graph and $N_\Delta(x)$ denotes the set of all neighbors of x in Δ. Following Habib and Maurer [3], a subset A of $V(\Delta)$ is externally related in Δ, if $N_\Delta(x) \setminus A = N_\Delta(y) \setminus A$, for all $x, y \in A$. Obviously, $\emptyset$, $\{x\}$, $x \in V(\Delta)$, and $V(\Delta)$ are externally related. The complete-empty $X-$join graph $\Gamma = (\biguplus_{x\in V(X)} X_x)_X$ is called reduced if for all vertices $x, y \in V(X)$ with the conditions that $x \neq y$ and $N_X(x) \setminus \{y\} = N_X(y) \setminus \{x\}$, the following hold:

(1) if $xy \notin E(X)$ then at least one of X_x and X_y is not an empty graph;

(2) if $xy \in E(X)$ then at least one of X_x and X_y is not a complete graph.

Consideration shall be given to the fact that K_1 can be regarded as both complete and empty. For an example, suppose Γ is an $X-$join graph in which $K_2 + K_1 + \Phi_2$ represents X and K_3 is a graph corresponding to each vertex of K_2 in X. Replacing vertices of Φ_2 of X by Φ_3 introduces Γ, completely. This $X-$join is not reduced. While as $\Gamma = K_6 + K_1 + \Phi_6$, one can observe that Γ is a P_2-join graph by corresponding Φ_6, K_1 and K_6 to vertices of P_2, respectively. So, considering Γ as the P_2-join is reduced.

Suppose Γ_1 and Γ_2 are graphs with disjoint vertex sets. The lexicographic product of Γ_1 and Γ_2 is another graph $\Gamma_1 o \Gamma_2$ with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1x_2 \in E(\Gamma_1)$ or $x_1 = x_2$ and $y_1y_2 \in E(\Gamma)$. Note that the lexicographic product is not commutative. If Γ is the $X-$join of graphs $\{X_x\}_{x\in V(X)}$ and $X_x \cong X_y$, for each $x, y \in V(X)$, then $\Gamma \cong X o X_x$, for some $x \in V(X)$.

Lemma 1.1. *Suppose x, y are vertices of a simple graph Δ. Then $N_\Delta(x) \setminus \{y\} = N_\Delta(y) \setminus \{x\}$ if and only if $\{x, y\}$ is externally related.*

Proof. It is easy to see that $N_\Delta(x) \setminus \{x, y\} \subseteq N_\Delta(x) \setminus \{y\}$ and $N_\Delta(y) \setminus \{x, y\} \subseteq N_\Delta(y) \setminus \{x\}$. Since Δ is simple, $N_\Delta(x) \setminus \{y\} \subseteq N_\Delta(x) \setminus \{x, y\}$ and $N_\Delta(y) \setminus \{x\} \subseteq N_\Delta(y) \setminus \{x, y\}$ proving the lemma. $\square$

The complete and empty graphs on a non-empty set B are denoted by K_B and Φ_B, respectively. In the case that $|B| = n$ we use the notation K_n as K_B and Φ_n as Φ_B. The complete bipartite graphs $K_{m,n}$ can be constructed as $K_{m,n} = \Phi_m + \Phi_n$. If Σ and Δ are graphs with $V(\Sigma) \subseteq V(\Delta)$ and $E(\Sigma) \subseteq E(\Delta)$, then we say Σ is a subgraph of Δ and write $\Sigma \leq \Delta$. If $T \subseteq V(\Delta)$ then the induced subgraph $\Delta[T]$ is a subgraph with $V(\Delta[T]) = T$ and $E(\Delta[T]) = \{e = uv \in E(\Delta) \mid \{u, v\} \subseteq T\}$. If C and D are subsets of $V(\Delta)$ and all elements of C are adjacent to all elements of D, then we write $C \sim D$. If there is no element in C to be adjacent with an element of D, then we use the notation $C \nsim D$. If Γ is a graph and $\mathcal{P}$ is a partition of $V(\Gamma)$ then the quotient graph $\frac{\Gamma}{\mathcal{P}}$ has the vertex set $\mathcal{P}$ and two partitions V_1 and V_2 are adjacent if there are $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1v_2 \in E(\Gamma)$. Our other notations are standard and can be taken from the standard book on graph theory.

The aim of this paper is to compute the automorphism group of the reduced complete-empty $X-$join graphs. To do this, we assume that Γ is such a graph. Choose X_x, $x \in V(X)$, to be the subgraph corresponding to the vertex x in X. Define $x \approx y$, if and only if $X_x \cong X_y$, where $x, y \in V(X)$. It is easy to see that $\approx$ is an equivalence relation. Moreover, we assume that T_x denotes the equivalence class of x under $\approx$ and W is a set of representatives for equivalence relation $\approx$. Define $\mathcal{A}(X) = \{f \in Aut(X) \mid \forall x \in W, f(T_x) = T_x\}$ as a subgroup of $Aut(X)$. Our main result is:

Theorem 1.2. *Suppose Γ is a reduced complete-empty $X-$join of graphs X_x, $x \in V(X)$. Then,*

$$Aut(\Gamma) \cong \left(\prod_{x \in V(X)} Sym(V(X_x))\right) \rtimes \mathcal{A}(X).$$

2. **Proof of the Main Theorem**

The aim of this section is to prove the main theorem of this paper.

Lemma 2.1. *Suppose Γ is an $X-$join and X_x is a graph corresponding to the vertex x of X. If $\sigma \in Aut(\Gamma)$ satisfies this condition that for each $x \in V(X)$, there exists $y \in V(X)$, such that $\sigma(X_x) = X_y$ then the function $f : V(X) \longrightarrow V(X)$ given by $f(x) = y$ is an automorphism of X.*

Proof. We assume that $t_x \in V(X_x)$, for each $x \in V(X)$. Then,

$$\begin{aligned} xx' \in E(X) &\Leftrightarrow t_x t_{x'} \in E(\Gamma) \\ &\Leftrightarrow \sigma(t_x)\sigma(t_{x'}) \in E(\Gamma) \\ &\Leftrightarrow yy' \in E(X) \\ &\Leftrightarrow f(x)f(x') \in E(X), \end{aligned}$$

where $y' = f(x')$, proving the lemma. □

Theorem 2.2. *Suppose* Γ *is a reduced complete-empty* $X-$*join and* X_x *is a graph corresponding to the vertex* x *of* X. *If for each* $y \in V(X)$, $X_x \cong X_y$, *then* $Aut(\Gamma) \cong Sym(V(Y)) \wr_{V(X)} Aut(X)$, *where* $Y \cong X_x$.

Proof. If $|V(X)| = 1, 2$ or $|V(X_x)| = 1$ then the proof will be clear. Hence we can assume that $|V(X)| \geq 3$ and $|V(X_x)| \geq 2$. Since for each $x \in V(X)$, X_x's are isomorphic and they are complete or empty, $Aut(X_x) \cong Sym(V(X_x))$. Define:

$$U = \{\sigma \in Sym(\Gamma) \mid \exists f \in Aut(X), \forall x \in V(X), \sigma(X_x) = X_{f(x)}\}.$$

We first prove that $U = Aut(\Gamma)$. To prove $U \subseteq Aut(\Gamma)$, we assume that $\sigma \in U$ and $a, b \in V(\Gamma)$. If there exists $x \in V(X)$, such that $a, b \in V(X_x)$ then there exists $f \in Aut(X)$, such that $\sigma(a), \sigma(b) \in \sigma(V(X_x)) = V(X_{f(x)})$. This shows that $ab \in E(\Gamma)$ if and only if $\sigma(a)\sigma(b) \in E(\Gamma)$. We now assume that there are $x, y \in V(X)$ such that $x \neq y$, $a \in V(X_x)$ and $b \in V(X_y)$. Then obviously there is an automorphism $f \in Aut(X)$ such that $\sigma(a) \in \sigma(X_x) = X_{f(x)}$, $\sigma(b) \in \sigma(X_y) = X_{f(y)}$. Therefore,

$$\begin{aligned} ab \in E(\Gamma) &\Leftrightarrow xy \in E(X) \\ &\Leftrightarrow f(x)f(y) \in E(X) \\ &\Leftrightarrow X_{f(x)} \sim X_{f(y)} \\ &\Leftrightarrow \sigma(a)\sigma(b) \in E(\Gamma). \end{aligned}$$

This proves that $U \subseteq Aut(\Gamma)$. To prove $Aut(\Gamma) \subseteq U$ we assume that $\theta \in Aut(\Gamma)$, $x \in V(X)$ and $a \in V(X_x)$. Since $|V(X_x)| \geq 2$, there exists a vertex $b \in V(X_x)$ such that $b \neq a$. There are two cases for $\theta(a)$ as follows:

(1) $\theta(a) \in V(X_x)$. We should prove that $\theta(b) \in V(X_x)$. Let us assume that, on the contrary, there exists $y \in V(X)$ such that $x \neq y$ and $\theta(b) \in V(X_y)$. If all X_x are complete then

$$ab \in E(X_x) \Rightarrow ab \in E(\Gamma) \Rightarrow \theta(a)\theta(b) \in E(\Gamma) \Rightarrow X_x \sim X_y.$$

If $|V(X)| = 2$ then clearly $N_X(x) \setminus \{y\} = N_X(y) \setminus \{x\} = \emptyset$ contradict by reducibility of Γ over X. Suppose $|V(X)| \geq 3$, $z \in N_X(x) \setminus \{y\}$ and $c \in X_z$. Then,

$$\begin{aligned} ac \in E(\Gamma) &\Leftrightarrow \theta(a)c \in E(\Gamma) && (\theta(a) \in V(X_x)) \\ &\Leftrightarrow a\theta^{-1}(c) \in E(\Gamma) \\ &\Leftrightarrow b\theta^{-1}(c) \in E(\Gamma) && (b \in V(X_x)) \\ &\Leftrightarrow \theta(b)c \in E(\Gamma) \\ &\Leftrightarrow z \in N_X(y) \setminus \{x\} && (\theta(b) \in V(X_y)). \end{aligned}$$

Hence $N_X(x)\setminus\{y\} = N_X(y)\setminus\{x\}$ which is impossible. If all X_x's are empty then there is no edge in X_x connecting a and b and so $\theta(a)$ is not adjacent to $\theta(b)$. So, there is no edge connecting X_x and X_y in Γ. Again, we can prove that $N_X(x)\setminus\{y\} = N_X(y)\setminus\{x\}$, a contradiction. So, $\theta(b) \in V(X_x)$.

(2) $\theta(a) \notin V(X_x)$. In this case there exists $y \in V(X)$ such that $\theta(a) \in V(X_y)$. We prove that $\theta(b) \in V(X_y)$. By the contrary, we assume that there exists $z \in V(X)$ such that $y \neq z$ and $\theta(b) \in V(X_z)$. If X_x's are complete then we have $ab \in E(X_x)$ which proves that $\theta(a)\theta(b) \in E(\Gamma)$. So, $X_y \sim X_z$. If $N_X(y) \setminus \{z\} = N_X(z) \setminus \{y\} = \emptyset$, then this is contradict by reducibility of Γ over X. If $t \in N_X(z) \setminus \{y\}$ and $c \in X_t$ then

$$\begin{aligned} tz \in E(X) &\Leftrightarrow c\theta(b) \in E(\Gamma) \\ &\Leftrightarrow \theta^{-1}(c)b \in E(\Gamma) \\ &\Leftrightarrow \theta^{-1}(c)a \in E(\Gamma) \\ &\Leftrightarrow c\theta(a) \in E(\Gamma) \\ &\Leftrightarrow ty \in E(X). \end{aligned}$$

This implies that $N_X(y)\setminus\{z\} = N_X(z)\setminus\{y\}$, contradict by the fact that Γ is a reduced complete-empty $X-$join. Now, we assume that all X_x's are empty. If $|V(X)| = 2$ then Γ is isomorphic to a complete bipartite graph and so $\theta(b) \in V(X_y)$, which is impossible. If $|V(X)| \geq 3$ then similar to the previous case, $N_X(y) \setminus \{z\} = N_X(z) \setminus \{y\}$ which is another contradiction. Thus, $\theta(b) \in V(X_y)$.

By above discussion, for each $x \in V(X)$, there exists a unique $y_x \in V(X)$ such that $\theta(X_x) = X_{y_x}$. Now, By defining $g(x) = y_x$ and Lemma 2.1, one can see that $\theta(X_x) = X_{g(x)}$. Therefore, $Aut(\Gamma) \subseteq U$ and

$$Aut(\Gamma) = \{\sigma \in Sym(\Gamma) \mid \exists f \in Aut(X), \forall x \in V(X), \sigma(X_x) = X_{f(x)}\}.$$

Define $\varphi : Aut(\Gamma) \to Aut(X)$ given by $\varphi(\sigma) = f_\sigma$, $\sigma \in Aut(\Gamma)$. Then, φ is an onto homomorphism such that for each $x \in V(X)$, $\sigma(X_x) = X_{f_\sigma(x)}$. Therefore,

$$\begin{aligned} Ker(\varphi) &= \{\sigma \in Aut(\Gamma) \mid \varphi(\sigma) = e_{Aut(X)}\} \\ &= \{\sigma \in Aut(\Gamma) \mid f_\sigma = e_{Aut(X)}\} \\ &= \{\sigma \in Aut(\Gamma) \mid \forall x \in V(X)\ ,\ \sigma(X_x) = X_x\} \\ &= \{\sigma \in Aut(\Gamma) \mid \sigma = \Pi_{x\in V(X)}\sigma_x\ s.t.\ \forall x \in V(X)\ ,\ \sigma_x \in Sym(V(X_x))\} \\ &= \prod_{x\in V(X)} Sym(V(X_x)) \cong \prod_{x\in V(X)} Sym(Y), \end{aligned}$$

By previous notations, $\Gamma \cong XoY$ and $\sigma_x \in Sym(V(Y))$. Without loss of generality, we can assume that $\sigma_x \in Sym(\{x\} \times V(Y))$. Set

$$B = \{\sigma_f \in Aut(XoY) \mid f \in Aut(X)\ ,\ \forall (x, y) \in V(XoY), \sigma_f((x, y)) = (f(x), y)\}.$$

It is clear that $B \cong Aut(X)$ and $B \cap Ker(\varphi) = \{\sigma_f \in B \mid \forall x \in V(X)\ ,\ f(x) = x\} = \{e_{Aut(\Gamma)}\}$. By definition of U and for each $(x, y) \in V(XoY)$, we have $\sigma((x, y)) = (f(x), \sigma_x(y)) = \sigma_f((x, \sigma_x(y))) = \sigma_f(\sigma_x((x, y))) = (\sigma_f o \sigma_x)((x, y))$. Therefore $\sigma \in BKer(\varphi)$ and so $Aut(\Gamma) = BKer(\varphi)$. Since $Ker(\varphi) \trianglelefteq Aut(\Gamma)$,

$$Aut(\Gamma) \cong Ker(\varphi) \rtimes B \cong \prod_{x\in V(X)} Sym(Y) \rtimes Aut(A) \cong Sym(Y) \wr_{V(X)} Aut(X).$$

This completes the proof. □

We mention here that Theorem 2.2 can be proved by [5, Theorem 3.1], but our proof is independent from some technical concepts like natural isomorphism, collapsed graph, section graph, $X-$subjoin, s-morphism and inverting $X-$point. On the other hand, three parts of the proof of Theorem 2.2 are needed to complete the proof of our main theorem.

Theorem 2.3. *Suppose Γ is a reduced complete-empty $X-$join and X_x is a graph corresponding to the vertex x of X. If for each distinct $x, y \in V(X)$, $X_x \ncong X_y$, then $Aut(\Gamma) \cong \prod_{x\in V(X)} Sym(V(X_x))$.*

Proof. If $|V(X)| = 1$ or 2 then the proof is trivial. Suppose $|V(X)| > 2$ and $\sigma \in Aut(\Gamma)$ is arbitrary. We prove that for each $x \in V(X)$, $\sigma(X_x) = X_x$. Since X_x's are non-isomorphic, for each $x \neq y \in V(X)$, $\sigma(X_x) \neq X_y$. Suppose that there are $x \in V(X)$ and $a \in V(X_x)$ with this property that $\sigma(a) \notin V(X_x)$. Hence there exists $y \in V(X)$, $y \neq x$, such that $\sigma(a) \in V(X_y)$. We consider four separate cases as follows:

(1) $X_x \cong K_1$. Suppose $V(X_x) = \{a\}$. Since X_s's are mutually non-isomorphic, $|V(X_y)| \geq 2$. So, there are $b \in V(X_y)$ and $z \in V(X)$, such that $b \neq \sigma(a)$ and $\sigma^{-1}(b) \in V(X_z)$. If $N_X(x) \neq \{z\}$, then we assume that $t \in N_X(x) \setminus \{z\}$ and $c \in V(X_t)$. Therefore,

$$\begin{aligned} tx \in E(X) &\Leftrightarrow ca \in E(\Gamma) && (c \in V(X_t), a \in V(X_x)) \\ &\Leftrightarrow \sigma(c)\sigma(a) \in E(\Gamma) \\ &\Leftrightarrow \sigma(c)b \in E(\Gamma) && (b, \sigma(a) \in V(X_y)) \\ &\Leftrightarrow c\sigma^{-1}(b) \in E(\Gamma) \\ &\Leftrightarrow tz \in E(X) && (\sigma^{-1}(b) \in V(X_z)). \end{aligned}$$

Hence $N_X(x) \setminus \{z\} = N_X(z) \setminus \{x\}$. If $N_X(x) = \{z\}$ then $N_X(x) \setminus \{z\} = N_X(z) \setminus \{x\} = \emptyset$. Since $|V(X_z)| \geq 2$, we can choose $d \neq \sigma^{-1}(b)$ in X_z. Then

$$\begin{aligned} b\sigma(a) \in E(\Gamma) &\Leftrightarrow \sigma^{-1}(b)a \in E(\Gamma) \\ &\Leftrightarrow zx \in E(X) && (\sigma^{-1}(b) \in V(X_z), a \in V(X_x)) \\ &\Leftrightarrow da \in E(\Gamma) && (d \in V(X_z)) \\ &\Leftrightarrow \sigma(d)\sigma(a) \in E(\Gamma) \\ &\Leftrightarrow \sigma(d)b \in E(\Gamma) && (b, \sigma(a) \in V(X_y)) \\ &\Leftrightarrow d\sigma^{-1}(b) \in E(\Gamma). \end{aligned}$$

This means that X_y is complete if and only if X_z is complete. If X_y is a complete graph, then $\sigma^{-1}(b)a \in E(\Gamma)$ and so $X_x \sim X_z$. But, $X_x \cong K_1$ which contradicts by reducibility of Γ over X. Thus, X_y is empty and we have $b\sigma(a) \notin E(\Gamma)$. Therefore, X_z is empty and $X_x \not\sim X_z$, which is impossible. This proves that $\sigma(X_x) = X_x$.

(2) $X_x \not\cong K_1$ and there exists $b \in V(X_x)$ such that $a \neq b$ and $\sigma(b) \in V(X_x)$. Note that X_x is complete if and only if $ab \in E(\Gamma)$. It means that $\sigma(a)\sigma(b) \in E(\Gamma)$ and so $X_x \sim X_y$. We first assume that $N_X(x) \neq \{y\}$. Suppose $z \in N_X(x) \setminus \{y\}$ and $c \in V(X_z)$. Then we have:

$$\begin{aligned} xz \in E(X) &\Leftrightarrow \sigma(b)c \in E(\Gamma) && (c \in V(X_z), \sigma(b) \in V(X_x)) \\ &\Leftrightarrow b\sigma^{-1}(c) \in E(\Gamma) \\ &\Leftrightarrow a\sigma^{-1}(c) \in E(\Gamma) && (a, b \in V(X_x)) \\ &\Leftrightarrow \sigma(a)c \in E(\Gamma) \\ &\Leftrightarrow yz \in E(X) && (\sigma(a) \in V(X_y)). \end{aligned}$$

So $N_X(x) \setminus \{y\} = N_X(y) \setminus \{x\}$. If $N_X(x) = \{y\}$ then again we have $N_X(x) \setminus \{y\} = N_X(y) \setminus \{x\} = \emptyset$. Since Γ is reduced, X_x is complete, if and only if X_y is an empty graph.

If $|V(X_y)| = 1$ then $X_y \cong K_1$ or $X_y \cong \Phi_1$, which leaded us to another contradiction. Therefore, there exists $d \in V(X_y)$ such that $d \neq \sigma(a)$. Hence,

$$\begin{aligned} xy \in E(X) &\Leftrightarrow \sigma(b)d \in E(\Gamma) && (\sigma(b) \in V(X_x), d \in V(X_y)) \\ &\Leftrightarrow b\sigma^{-1}(d) \in E(\Gamma) \\ &\Leftrightarrow a\sigma^{-1}(d) \in E(\Gamma) && (a, b \in V(X_x)) \\ &\Leftrightarrow \sigma(a)d \in E(\Gamma). \end{aligned}$$

It means that X_x is complete, if and only if X_y is complete which is a contradiction. Therefore, for each $x \in V(X)$, $\sigma(X_x) = X_x$.

(3) $X_x \ncong K_1$ and there exists $b \in V(X_x)$ such that $b \neq a$ and $\sigma(b) \notin V(X_x) \cup V(X_y)$. In this case, there exists $z \in V(X_x)$ such that $z \neq x, y$ and $\sigma(b) \in V(X_z)$. X_x is complete if and only if ab is an edge of Γ and so $\sigma(a)\sigma(b) \in E(\Gamma)$. It means that X_x is a complete graph if and only if $yz \in E(X)$. By our assumption $N_X(z) \neq \{y\}$. Set $t \in N_X(z) \setminus \{y\}$ and $c \in V(X_t)$. Since

$$\begin{aligned} tz \in E(X) &\Leftrightarrow c\sigma(b) \in E(\Gamma) && (c \in V(X_t), \sigma(b) \in V(X_z)) \\ &\Leftrightarrow \sigma^{-1}(c)b \in E(\Gamma) \\ &\Leftrightarrow \sigma^{-1}(c)a \in E(\Gamma) && (a, b \in V(X_x)) \\ &\Leftrightarrow c\sigma(a) \in E(\Gamma) \\ &\Leftrightarrow ty \in E(X) && (\sigma(a) \in V(X_y)), \end{aligned}$$

Thus, $N_X(y) \setminus \{z\} = N_X(z) \setminus \{y\}$. If $N_X(z) = \{y\}$ then again $N_X(y) \setminus \{z\} = N_X(z) \setminus \{y\} = \emptyset$. Since Γ is reduced, exactly one of X_y or X_z are empty graph. On the other hand, X_s's are mutually non-isomorphic and so none of X_y or X_z have one vertex. Choose $d \in V(X_y)$ and $e \in V(X_z)$. Then,

$$\begin{aligned} \sigma(a)d \in E(\Gamma) &\Leftrightarrow a\sigma^{-1}(d) \in E(\Gamma) \\ &\Leftrightarrow b\sigma^{-1}(d) \in E(\Gamma) && (a, b \in V(X_x)) \\ &\Leftrightarrow \sigma(b)d \in E(\Gamma) \\ &\Leftrightarrow e\sigma(a) \in E(\Gamma) && (\sigma(b), e \in V(X_z), d, \sigma(a) \in V(X_y)) \\ &\Leftrightarrow \sigma^{-1}(e)a \in E(\Gamma) \\ &\Leftrightarrow \sigma^{-1}(e)b \in E(\Gamma) \\ &\Leftrightarrow e\sigma(b) \in E(\Gamma). \end{aligned}$$

This shows that X_y is complete if and only if X_z is a complete graph which is our final contradiction. So, $\sigma(X_x) = X_x$.

(4) $X_x \ncong K_1$ and there exists $b \in V(X_x)$ such that $a \neq b$ and $\sigma(b) \in V(X_y)$. Since $ab \in E(\Gamma)$ if and only if X_x is complete, one can easily see that X_x is a complete graph, if and only if X_y is complete. If $|V(X_x)| = |V(X_y)| = 2$ then $X_x \cong X_y$, which is impossible. If $|V(X_y)| = 2$ then $|V(X_x)| \geq 3$ and so there exists $c \in V(X_x)$ such that $\sigma(c) \notin V(X_y)$. But, this is the case (3) which is a contradiction. Thus $|V(X_y)| \geq 3$. Then there exists an element $c \in V(X_y)$ such that $c \neq \sigma(a), \sigma(b)$. But there is $z \in V(X)$, such that $\sigma^{-1}(c) \in V(X_z)$. We assume that $N_X(x) \setminus \{z\} \neq \emptyset$, $t \in N_X(x) \setminus \{z\}$ and $d \in V(X_t)$. Then,

$$\begin{aligned} tx \in E(X) &\Leftrightarrow da \in E(\Gamma) && (d \in V(X_t), a \in V(X_x)) \\ &\Leftrightarrow \sigma(d)\sigma(a) \in E(\Gamma) \\ &\Leftrightarrow \sigma(d)c \in E(\Gamma) && (\sigma(a), c \in V(X_y)) \\ &\Leftrightarrow d\sigma^{-1}(c) \in E(\Gamma) \\ &\Leftrightarrow tz \in E(X) && (\sigma^{-1}(c) \in V(X_z)). \end{aligned}$$

Therefore, $N_X(x) \setminus \{z\} = N_X(z) \setminus \{x\}$. If X_y is complete then $\sigma(a)c$ is an edge of Γ, which implies that $a\sigma^{-1}(c) \in E(\Gamma)$. Ultimately, $xz \in E(X)$. By a similar argument, we can see that if $xz \in E(X)$, then the graph X_y is complete. Now, the reducibility of Γ over X concludes that X_x is complete, if and only if X_z is an empty graph. Hence, $|V(X_z)| \geq 2$ and so, there exists $e \in V(X_z)$ such that $e \neq \sigma^{-1}(c)$. Hence,

$$\begin{aligned} ab \in E(\Gamma) &\Leftrightarrow \sigma(a)\sigma(b) \in E(\Gamma) \\ &\Leftrightarrow \sigma(a)c \in E(\Gamma) && (\sigma(b), c \in V(X_y)) \\ &\Leftrightarrow a\sigma^{-1}(c) \in E(\Gamma) \\ &\Leftrightarrow be \in E(\Gamma) && (a, b \in V(X_x), \sigma^{-1}(c), e \in V(X_z)) \\ &\Leftrightarrow \sigma(b)\sigma(e) \in E(\Gamma) \\ &\Leftrightarrow c\sigma(e) \in E(\Gamma) \\ &\Leftrightarrow \sigma^{-1}(c)e \in E(\Gamma). \end{aligned}$$

Above argument shows that X_x is complete if and only if X_z is a complete graph, which is our final contradiction. So, we have again $\sigma(X_x) = X_x$, as desired.

Therefore, each $\sigma \in Aut(\Gamma)$ has a decomposition into σ_x's, where $\sigma_x \in Aut(X_x)$. Thus, $Aut(\Gamma)$ can be written as an inner product of $Aut(X_x)$'s. Furthermore, for every $x, y \in V(X)$, $y \neq x$, we have $Aut(X_x) \cap Aut(X_y) = \{e_{Aut(\Gamma)}\}$. Since X_x's are complete or empty, $Aut(X_x) \cong Sym(V(X_x))$. Therefore, $Aut(\Gamma) \cong \prod_{x \in V(X)} Sym(V(X_x))$, that completes our proof. □

Proof of the Main Theorem. It is clear that if $|V(X)| = 1$ or $|V(X_x)| = 1$, $x \in V(X)$, then the proof is trivial. Hence it is enough to investigate the case that $|V(X)|, |V(X_x)| \geq 2$. Since all X_x's are complete or empty, $Aut(X_x) \cong Sym(V(X_x))$, for each $x \in V(X)$. Define

$$C = \{\sigma \in Sym(\Gamma) \mid \exists f \in \mathcal{A}(X), \forall x \in V(X), \sigma(X_x) = X_{f(x)}\}.$$

We prove that $C = Aut(\Gamma)$. Let $\sigma \in C$ and $a, b \in V(\Gamma)$. If there exists $x \in V(X)$ such that $a, b \in V(X_x)$, then $\sigma(a), \sigma(b) \in V(\sigma(X_x)) = V(X_{f(x)})$, for some $f \in Aut(X)$ that $f(T_x) = T_x$. Since $X_x \cong X_{f(x)}$, $ab \in V(\Gamma)$ if and only if $\sigma(a)\sigma(b) \in E(\Gamma)$. This implies that $C \subseteq Aut(\Gamma)$. If there exists $x, y \in V(X)$, with this property that $x \neq y$, $a \in V(X_x)$ and $b \in V(X_y)$, then by applying similar arguments as Theorem 2.2 we can see that $C \subseteq Aut(\Gamma)$. Conversely, suppose $\theta \in Aut(\Gamma)$, $x \in V(X)$ and $a \in V(X_x)$. We show that there exists $h \in \mathcal{A}(X)$ such that $\theta(X_x) = X_{h(x)}$. If $|V(X_x)| = 1$ then by the proof of Theorem 2.3 (a), $\theta(X_x) = X_y$ in which $y \in T_x$. We proceed the proof by assuming that $|V(X_x)| \geq 2$. Choose b, $b \neq a$, to be an arbitrary vertex of X_x. We have three separate cases as follows:

a. $\theta(a) \in V(X_x)$. In this case, if $\theta(b) \in V(X_y)$, $y \in T_x$, then by a similar argument as the proof of Theorem 2.2 (a), $x = y$. Furthermore, if $\theta(b) \in V(X_z)$, for some $z \in V(X)$ such that $z \notin T_x$, then by the proof of Theorem 2.3 (b), we lead to a contradiction. So, $\theta(b) \in V(X_x)$.

b. $\theta(a) \in V(X_y)$ such that $y \in T_x$. In this case, the argument of the previous case applies for $x = y$. If $\theta(b) \in V(X_z)$, $z \in T_x$, then a similar argument as the proof of Theorem 2.2 (b), lead us to $y = z$. We now assume that $\theta(b) \in V(X_t)$, for some $t \in V(X)$ such that $t \notin T_x$. It is easy to see that X_x is complete if and only if $ab \in E(\Gamma)$ and also, $yz \in E(X)$ if and only if $\theta(a)\theta(b) \in E(\Gamma)$. So, X_x is a complete graph if and only if $yz \in E(X)$. Since X_x and X_y are isomorphic, X_y is complete if and only if $yz \in E(X)$. Note that if $N_X(y) \setminus \{z\} = \emptyset$ then $N_X(y) \setminus \{z\} = N_X(z) \setminus \{y\}$. Suppose $N_X(y) \setminus \{z\} \neq \emptyset$, $t \in N_X(y) \setminus \{z\}$ and $c \in V(X_t)$. Then,

$$\begin{aligned} ty \in E(X) &\Leftrightarrow c\theta(a) \in E(\Gamma) \\ &\Leftrightarrow \theta^{-1}(c)a \in E(\Gamma) \\ &\Leftrightarrow \theta^{-1}(c)b \in E(\Gamma) \\ &\Leftrightarrow c\theta(b) \in E(\Gamma) \\ &\Leftrightarrow tz \in E(X). \end{aligned}$$

By above relations and the fact that Γ is a reduced complete-empty $X-$join, X_x and X_y are complete graphs if and only if X_z is empty. On the other hand, by the following relations and the fact that X_z is an empty graph if and only if $d\theta(b) \notin E(\Gamma)$, one can deduce that X_y is complete if and only if $yz \notin E(X)$, which is a contradiction:

$$\begin{aligned} d\theta(b) \notin E(\Gamma) &\Leftrightarrow \theta^{-1}(d)b \notin E(\Gamma) \\ &\Leftrightarrow \theta^{-1}(d)a \notin E(\Gamma) \\ &\Leftrightarrow d\theta(a) \notin E(\Gamma) \\ &\Leftrightarrow yz \notin E(X). \end{aligned}$$

c. $\theta(a) \in V(X_z)$ and $z \notin T_x$. If $(\theta(b) \in E(X))$ or $(\theta(b) \in V(X_t)$ and $t \in T_x)$ or $(\theta(b) \in V(X_s)$ and $s \notin T_x)$ then the proof of Theorem 2.3 (b), the Case (b) or Theorem 2.3 (c) leaded us to a contradiction. Therefore, $\theta(b) \in V(X_z)$.

We observe in three cases that for each $x \in V(X)$, there exists $y_x \in V(X)$ such that $\theta(X_x) = X_{y_x}$. By putting $h(x) = y_x$ and Lemma 2.1, we have $\theta(X_x) = X_{h(x)}$. This proves that $C = Aut(\Gamma)$. Since the map $\varphi : Aut(\Gamma) \to \mathcal{A}(X)$ given by $\varphi(\sigma) = f_\sigma$ is an onto group homomorphism and for every $x \in V(X)$, $\sigma(X_x) = X_{f_\sigma(x)}$,

$$\begin{aligned} Ker(\varphi) &= \{\sigma \in Aut(\Gamma) \mid \varphi(\sigma) = e_{\mathcal{A}(X)}\} \\ &= \{\sigma \in Aut(\Gamma) \mid f_\sigma = e_{\mathcal{A}(X)}\} \\ &= \{\sigma \in Aut(\Gamma) \mid \forall x \in V(X)\ ,\ \sigma(X_x) = X_x\} \\ &= \{\sigma \in Aut(\Gamma) \mid \sigma = \Pi_{x\in V(X)}\sigma_x\ s.t.\ \sigma_x \in Sym(V(X_x))\} \\ &= \prod_{x\in V(X)} Sym(V(X_x)). \end{aligned}$$

We use the notation (x, a) for vertices of X_a in Γ. Without loss of generality, we can assume that $\sigma_x \in Sym(\{x\} \times V(X_x))$. If $f \in \mathcal{A}(X)$, then the mapping σ_f defined as $\sigma_f((x, a)) = (f(x), a)$ is an automorphism of Γ. Define D to be the set of all such automorphisms. It is clear that $D \cong \mathcal{A}(X)$ and $D \cap Ker(\varphi) = \{\sigma_f \in D \mid \forall x \in V(X)\ ,\ f(x) = x\} = \{e_{Aut(\Gamma)}\}$. Suppose $\sigma \in Aut(\Gamma)$. Then, there exists $f \in \mathcal{A}(X)$ such that for each $x \in V(X)$, $\sigma(X_x) = X_{f(x)}$. Furthermore, for every $(x, a) \in V(\Gamma)$, $\sigma((x, a)) = (f(x), \sigma_x(a)) = \sigma_f((x, \sigma_x(a))) = \sigma_f(\sigma_x((x, a))) = (\sigma_f o \sigma_x)((x, a))$. Therefore, $\sigma = \sigma_f o \sigma_x \in DKer(\varphi)$ which shows that $Aut(\Gamma) = DKer(\varphi)$. Since $Ker(\varphi)$ is normal in $Aut(\Gamma)$,

$$Aut(\Gamma) \cong Ker(\varphi) \rtimes D \cong (\prod_{x\in V(X)} Sym(V(X_x))) \rtimes \mathcal{A}(X).$$

This completes our argument. □

Corollary 2.4. *Suppose Γ is a reduced complete-empty $X-$join and X_x denotes the subgraph corresponding to the vertex $x \in V(X)$. Then $Aut(X_x)$ is isomorphic to a normal subgroup of $Aut(\Gamma)$.*

Proof. Set $S_x = \{\sigma \in Aut(\Gamma) \mid \forall a \in V(\Gamma) \setminus V(X_x), \sigma(a) = a\}$, where $x \in V(X)$. It is clear that $S_x \cong Sym(V(X_x))$ and $S_x \leq Aut(\Gamma)$. Let σ and θ be arbitrary elements of $Aut(\Gamma)$ and S_x, respectively. By the proof of Theorem 1.2, for each $y \in V(X)$, there exists $z_y \in V(X)$, such that $\sigma(X_y) = X_{z_y}$. So, $\sigma^{-1}\theta\sigma(X_y) = \sigma^{-1}\theta(X_{z_y}) = \sigma^{-1}(X_{z_y}) = X_y$, which shows that S_x is isomorphic to a normal subgroup of $Aut(\Gamma)$. □

3. Applications

Suppose Γ is a connected graph. If the intersection of each decreasing chain of neighborhoods of vertex subsets are non-empty then Γ is called an NDC graph. For example, every finite graph or infinite graph in which its vertices have finite degrees satisfies NDC condition. It is far from true that each graph is NDC. To see this, it is enough to check the graph Λ with $V(\Lambda) = \mathbb{R}$ and $E(\Lambda) = \{xy \mid |xy| < 1\}$. For each $i \in \mathbb{N}$, define $A_i = \{i\}$. Then $N(A_i) = (-\frac{1}{i}, \frac{1}{i})$, $N(A_{i+1}) \subseteq N(A_i)$ and $\bigcap_{i\in\mathbb{N}} N(A_i) = \emptyset$. Therefore, Λ is not an NDC graph. In this section, we apply our results in Section 2 to obtain the main properties of connected NDC graphs.

Theorem 3.1. *Let Γ be a simple connected graph which satisfies NDC condition. Then Γ can be written as a reduced complete-empty $X-$join.*

Proof. If Γ is complete, then Γ can be written as a reduced complete K_1-join. So, we can assume that Γ is not complete. Define the sets NN, CE and CE_m as follows:

$$\begin{aligned} NN &= \{U \subseteq V(\Gamma) \mid N(U) \neq \emptyset\}, \\ CE &= \{U \in NN \mid \Gamma[U] \text{ is complete or empty graph}\}, \\ CE_m &= \{U \in CE \mid \forall x, y \in U, N(x) \setminus \{y\} = N(y) \setminus \{x\}\}. \end{aligned}$$

The sets NN, CE and CE_m are not empty, since they are containing single points. It is clear that CE_m is a partially ordered set under set inclusion. We now prove that for each $a \in V(\Gamma)$, CE_m has a maximal element containing a. Define the chain $\{U_i\}_{i\in I}$, for each non-empty and arbitrary set I, such that one of the members of this chain is $\{a\}$. We claim that $\bigcup_{i\in I} U_i \in CE_m$. If for all $i \in I$, $U_i = \{a\}$, then our claim is trivial. Suppose $\bigcup_{i\in I} U_i \neq \{a\}$ and $\mathfrak{U} = \bigcup_{i\in I} U_i$. We first prove that $N(\mathfrak{U}) \neq \emptyset$. Note that $x \in N(\bigcup_{i\in I} U_i)$ if and only if for all $i \in I, x \in N(U_i)$ if and only if $x \in \bigcap_{i\in I} N(U_i)$. This proves that $N(\bigcup_{i\in I} U_i) = \bigcap_{i\in I} N(U_i)$. Now NDC condition implies that $\bigcap_{i\in I} N(U_i) \neq \emptyset$ and so $N(\mathfrak{U}) \neq \emptyset$.

We now prove that $\Gamma[\mathfrak{U}]$ is complete or empty. We first assume that there exists $l \in I$ such that $\Gamma[U_l]$ is complete and non-isomorphic to K_1. For each $i \in I$ with $U_l \subseteq U_i$, $\Gamma[U_i]$ is a complete graph and so $\Gamma[\mathfrak{U}]$ is complete. If for each $i \in I$, $\Gamma[U_l]$ is empty then it is clear that $\Gamma[\mathfrak{U}]$ is also empty. Therefore, we observe that in each case $\Gamma[\mathfrak{U}]$ is complete or empty.

Choose elements x and y in $\mathfrak{U}$, then there are $j,k \in I$ such that $x \in U_j$ and $y \in U_k$. Since U_i's are chain, without loss of generality, we can assume that $U_j \subseteq U_k$. Thus $x,y \in U_k$. Since $U_k \in CE_m$, $N(x) \setminus \{y\} = N(y) \setminus \{x\}$ which proves that $\mathfrak{U} \in CE_m$. It can easily see that $\mathfrak{U}$ is a maximal member of this chain and so by Zorn's lemma, CE_m has a maximal element M in the set $\{T \in CE_m \mid a \in T\}$. The maximal member M is also the maximal member of CE_m, since otherwise, M is contained in a member of CE_m, contradicts by maximality of M.

The set of all maximal elements of CE_m is denoted by $\mathcal{M}$. Since CE_m is containing all singletons, $\bigcup \mathcal{M} = \bigcup CE_m = V(\Gamma)$. To prove that the intersection of two arbitrary different elements of $\mathcal{M}$ is empty, we assume by contrary that $M_1, M_2 \in \mathcal{M}$ such that $M_1 \cap M_2 \neq \emptyset$. It is now easy to prove that $|M_1|, |M_2| \neq 1$. Our proof will consider two cases as follows:

a. $|M_1 \cap M_2| \geq 2$. We first notice that $\Gamma[M_1]$ is complete if and only if $\Gamma[M_1 \cap M_2]$ is complete if and only if $\Gamma[M_2]$ is complete and in the same way $\Gamma[M_1]$ is empty if and only if $\Gamma[M_2]$ is an empty graph. Without loss of generality, we assume that $M_1 \setminus M_2$ is nonempty and let $y \in M_1 \setminus M_2$. We now assume that $\Gamma[M_1]$ and $\Gamma[M_2]$ are complete graphs and $x \in M_1 \cap M_2$. By assumption $N_\Gamma(x) \setminus \{y\} = N_\Gamma(y) \setminus \{x\}$. If $M_2 \setminus M_1$ is a nonempty set, then all of its elements are adjacent to x in Γ and we have $M_2 \setminus M_1 \subseteq N_\Gamma(y) \setminus \{x\}$. This implies that $\Gamma[M_1 \cup M_2]$ is complete which contradicts by maximality of M_1 and M_2 in CE_m. Hence $M_2 \setminus M_1 = \emptyset$ and so $M_2 \subsetneq M_1$. This leaded us to another contradiction by maximality of M_2. Next we assume that $\Gamma[M_1]$ and $\Gamma[M_2]$ are both empty graphs. Choose $x \in M_1 \cap M_2$. Since $N_\Gamma(x) \setminus \{y\} = N_\Gamma(y) \setminus \{x\}$, $N_\Gamma(x) = N_\Gamma(y)$. Moreover, if $z \in M_2 \setminus M_1$, then $N_\Gamma(x) \setminus \{z\} = N_\Gamma(z) \setminus \{x\}$ and so $N_\Gamma(x) = N_\Gamma(z)$. Therefore, $N_\Gamma(x) = N_\Gamma(y) = N_\Gamma(z)$ which implies that $M_1 \cup M_2 \in CE_m$, a contradiction. If $M_2 \setminus M_1 = \emptyset$, then again $M_2 \subsetneq M_1$ and this is our final contradiction for this case.

b. $|M_1 \cap M_2| = 1$. If both of $\Gamma[M_1]$ and $\Gamma[M_2]$ are complete or empty graphs, then a similar argument as case (a), leaded us to a contradiction. Thus, without loss of generality, we can assume that $\Gamma[M_1]$ is a complete graph and $\Gamma[M_2]$ is empty. Choose $x \in M_1 \cap M_2$, $y \in M_1 \setminus M_2$ and $z \in M_2 \setminus M_1$. By definition of CE_m, $N_\Gamma(x) \setminus \{z\} = N_\Gamma(z) \setminus \{x\}$. Since $y \in N_\Gamma(x) \setminus \{z\}$, $y \in N_\Gamma(z) \setminus \{x\}$ which shows that $z \in N_\Gamma(y) \setminus \{x\}$. On the other hand, $N_\Gamma(x) \setminus \{y\} = N_\Gamma(y) \setminus \{x\}$ implies that $z \in N_\Gamma(x) \setminus \{y\}$ which is impossible.

In each case we lead to a contradiction which shows that $\mathcal{M}$ is a partition of vertices of Γ. Suppose M and M' are arbitrary distinct elements of $\mathcal{M}$, $x \in M$ and $x' \in M'$. We will prove that $xx' \in E(\Gamma)$ if and only if $\Gamma[M] \sim \Gamma[M']$. It is enough to assume that $xx' \in E(\Gamma)$, $y \in M$ and $y' \in M'$. Since $x' \in N_\Gamma(x)$, $x' \in N_\Gamma(y)$ and so $x'y \in E(\Gamma)$, which proves that x' is adjacent to all elements of M. A similar argument shows that x is adjacent to all elements of M'. Since $y \in N_\Gamma(x') \setminus \{y'\}$ and $N_\Gamma(x') \setminus \{y'\} = N_\Gamma(y') \setminus \{x'\}$, y and y' are adjacent, which implies that all elements of M are adjacent to all elements of M'. Therefore, $\Gamma[M] \sim \Gamma[M']$ and $xx' \notin E(\Gamma)$ if and only if there is no an edge that connects a vertex in $\Gamma[M]$ to another vertex in $\Gamma[M']$. Define the graph X with vertex set $V(X) = \mathcal{M}$ and edge set $E(X) = \{M_1M_2 \mid \Gamma[M_1] \sim \Gamma[M_2]\}$ and $\Gamma[M]$ is the graph corresponding to the vertex M in X. Since any subgraph generated by elements of CE_m are complete or empty, the subgraph generated by all elements of $\mathcal{M}$ is also complete or empty. Thus, $\Gamma[M]$ is complete or empty. This shows that the graph Γ can be written as an X-join of complete or empty graphs. We now prove that this join is reduced. To do this, we consider two arbitrary vertices M_1 and M_2 such that $N_X(M_1) \setminus \{M_2\} = N_X(M_2) \setminus \{M_1\}$.

We first assume that $M_1M_2 \notin E(X)$ and show that at least one of the graphs $\Gamma[M_1]$ or $\Gamma[M_2]$ are not empty. By contrary, suppose both of $\Gamma[M_1]$ and $\Gamma[M_2]$ are empty graphs. By connectedness of Γ, there exists $M_3 \in V(X)$ such that $M_3 \neq M_1, M_2$ and $M_1M_3, M_2M_3 \in E(X)$. It is clear that $M_3 \subseteq N(M_1 \cup M_2)$ and so $N(M_1 \cup M_2) \neq \emptyset$. Since M_1 and M_2 are empty, $M_1 \cup M_2$ is empty and hence $M_1 \cup M_2 \in CE$. Choose arbitrary distinct elements $x_1, x_2 \in M_1 \cup M_2$. If $x_1, x_2 \in M_1$ or $x_1, x_2 \in M_2$, then one can easily see that $N_\Gamma(x_1) \setminus \{x_2\} = N_\Gamma(x_2) \setminus \{x_1\}$, which shows that $M_1 \cup M_2 \in CE_m$. This is a contradiction by maximality of M_1 and M_2. Therefore, without loss of generality, we can assume that $x_1 \in M_1$ and $x_2 \in M_2$. Obviously, for each $a \in N_\Gamma(x_1) \setminus \{x_2\}$, there exists $M_a \in V(X)$ such that $a \in M_a$ and $M_a \neq M_2$. Since $ax_1 \in E(\Gamma)$, $M_a \neq M_1$ and so $M_1M_a \in E(X)$. Hence $M_a \in N_X(M_2) \setminus \{M_1\}$, which implies that $ax_2 \in E(\Gamma)$. Thus $a \in N_\Gamma(x_2) \setminus \{x_1\}$, which shows that $N_\Gamma(x_1) \setminus \{x_2\} \subseteq N_\Gamma(x_2) \setminus \{x_1\}$. By a similar argument as above, we can see that $N_\Gamma(x_2) \setminus \{x_1\} \subseteq N_\Gamma(x_1) \setminus \{x_2\}$, which proves $N_\Gamma(x_1) \setminus \{x_2\} = N_\Gamma(x_2) \setminus \{x_1\}$. Therefore, $M_1 \cup M_2 \in CE_m$, which is impossible. This contradiction shows that at least one of $\Gamma[M_1]$ and $\Gamma[M_2]$ are nonempty.

We now assume that $M_1M_2 \in E(X)$ and prove at least one of the graphs $\Gamma[M_1]$ and $\Gamma[M_2]$ are not complete. To see this, we assume by contrary that both of them are complete graphs. If $N_X(M_1) \setminus \{M_2\}$ is empty, then $X \cong K_2$ and graphs corresponding to two vertices of K_2 are complete which concludes that Γ is complete. This lead us to another contradiction. Therefore, $N_X(M_1) \setminus \{M_2\} = N_X(M_2) \setminus \{M_1\} \neq \emptyset$. Therefore, there exists $M_4 \in V(X)$ such that $M_4M_1, M_4M_2 \in E(X)$. Hence $M_4 \subseteq N(M_1 \cup M_2)$ and so $N(M_1 \cup M_2) \neq \emptyset$. Since $\Gamma[M_1]$ and $\Gamma[M_2]$ are complete and $M_1M_2 \in E(X)$, $\Gamma[M_1 \cup M_2]$ is complete and so $M_1 \cup M_2 \in CE$. Choose $y_1, y_2 \in M_1 \cup M_2$. If $y_1, y_2 \in M_1$ or $y_1, y_2 \in M_2$, then we can easily see that $N_\Gamma(y_1) \setminus \{y_2\} = N_\Gamma(y_2) \setminus \{y_1\}$. So, $M_1 \cup M_2 \in CE_m$ that lead us to a contradiction. Then, without loss of generality, we can assume that $y_1 \in M_1$ and $y_2 \in M_2$. Choose $a \in N_\Gamma(y_1) \setminus \{y_2\}$. If $a \in M_1$ then $ay_2 \in E(\Gamma)$ and so $a \in N_\Gamma(y_2) \setminus \{y_1\}$. If $a \in M_2$, then $ay_2 \in E(\Gamma)$ which again proves that $a \in N_\Gamma(y_2) \setminus \{y_1\}$. If $a \notin M_1 \cup M_2$, then there are $M_3 \in V(X)$, $M_3 \neq M_1, M_2$, such that $a \in M_3$. Since $ax \in E(\Gamma)$, $M_3M_1 \in E(X)$ and so $M_3 \in N_X(M_1) \setminus \{M_2\}$. This concludes that $M_3 \in N_X(M_2) \setminus \{M_1\}$. Hence $M_3M_2 \in E(X)$ and $a \in N_\Gamma(y_2) \setminus \{y_1\}$. Therefore, $N_\Gamma(y_1) \setminus \{y_2\} \subseteq N_\Gamma(y_2) \setminus \{y_1\}$ and similarly $N_\Gamma(y_2) \setminus \{y_1\} \subseteq N_\Gamma(y_1) \setminus \{y_2\}$. This leaded us to $M_1 \cup M_2 \in CE_m$ that contradicts by maximality of M_1 and M_2, proving the result. □

We are now ready to investigate the case of non-locally infinite NDC graphs. Suppose $\Gamma = K_A + \Phi_B$ in which A and B are infinite sets. It is trivial that Γ is a non-locally infinite NDC graph and it can be written as a reduced K_2-join of K_A and Φ_B. By previous theorem,

$$\begin{aligned} NN &= \{U \subseteq V(\Gamma) \mid A \cup C \nsubseteq U \ , \ \emptyset \neq C \subseteq B\}, \\ CE &= \{U \in NN \mid U \subseteq A \text{ or } U \subseteq B\}, \\ CE_m &= CE. \end{aligned}$$

It is obvious that $\mathcal{M} = \{A, B\}$. Define X to be a graph with $V(X) = \{A, B\}$ and $E(X) = \{AB\}$. Then, Γ is a reduced K_2- join of $\Gamma[A] = K_A$ and $\Gamma[B] = \Phi_B$.

Lemma 3.2. *Let Γ be an $X-$join graph. Then, there exists a partition $\mathcal{P}$ of $V(\Gamma)$ such that $\Gamma/\mathcal{P} \cong X$.*

Proof. Suppose X_x is a graph corresponding to the vertex $x \in X$. Define $P_x = V(X_x)$ and let $\mathcal{P} = \{P_x \mid x \in V(X)\}$. It is clear that the function $f : \Gamma/\mathcal{P} \to X$ given by $f(P_x) = x$ is a bijection. To complete the proof, it should be proved that $P_xP_y \in E(\Gamma/\mathcal{P})$ if and only if $xy \in E(X)$, for every $x, y \in V(X)$. To do this, we note that $P_xP_y \in E(\Gamma/\mathcal{P})$ if and only if there are $p_x \in P_x$ and $p_y \in P_y$ such that $p_xp_y \in E(\Gamma)$ if and only if there are $p_x \in V(X_x)$ and $p_y \in V(X_y)$ such that $p_xp_y \in E(\Gamma)$ if and only if $xy \in E(X)$. Hence the result. □

In the following theorem, it is proved that the graph X in Theorem 3.1 is unique.

Theorem 3.3. *Let Γ be a reduced complete-empty $X-$ and $Y-$join. Then $X \cong Y$.*

Proof. By Lemma 3.2, $V(\Gamma)$ have two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\Gamma/\mathcal{P}_1 \cong X$ and $\Gamma/\mathcal{P}_2 \cong Y$. If $\mathcal{P}_1 = \mathcal{P}_2$ then clearly $X \cong Y$. Suppose $\mathcal{P}_1 \neq \mathcal{P}_2$. Without loss of generality we assume that there exists $P_1 \in \mathcal{P}_1 \setminus \mathcal{P}_2$. Then there exist $P_2, P_2' \in \mathcal{P}_2$ such that $P_1 \cap P_2, P_1 \cap P_2' \neq \emptyset$. Suppose $y_2, y_2' \in V(Y)$ are corresponding to the graphs $\Gamma[P_2]$ and $\Gamma[P_2']$, respectively. We also assume that $y \in N_Y(y_2) \setminus \{y_2'\}$ and denote the graph corresponding to y by Y_y. So, there exists $P_2'' \in \mathcal{P}_2$ such that $Y_y = \Gamma[P_2'']$. We will consider two separate cases as follows:

a. $P_2'' \setminus P_1 \neq \emptyset$. Suppose $a \in P_2'' \setminus P_1$. Hence there exists $P_1' \in \mathcal{P}_1$ such that $a \in P_1'$. Since a is adjacent to all vertices of $\Gamma[P_2]$ and $P_1 \cap P_2 \neq \emptyset$, all elements of P_1 are adjacent to all elements of P_1'. Thus, a is adjacent to all elements of $P_1 \cap P_2'$ and so $\Gamma[P_2''] \sim \Gamma[P_2']$. Hence, $y \in N_Y(y_2') \setminus \{y_2\}$ and $N_Y(y_2) \setminus \{y_2'\} \subseteq N_Y(y_2') \setminus \{y_2\}$, as desired.

b. $P_2'' \subseteq P_1$. Since $\Gamma[P_2] \sim \Gamma[P_2'']$, all elements of P_2'' are adjacent to all elements of $P_1 \cap P_2'$. Hence, $E(\Gamma[P_1]) \neq \emptyset$ and $\Gamma[P_1]$ is complete. This concludes that all elements of P_2'' are adjacent to all elements of $P_1 \cap P_2'$ and so, they are adjacent to vertices of $\Gamma[P_2']$. Since $y \in N_Y(y_2') \setminus \{y_2\}$, $N_Y(y_2) \setminus \{y_2'\} \subseteq N_Y(y_2') \setminus \{y_2\}$.

Note that $N_Y(y_2') \setminus \{y_2\} \subseteq N_Y(y_2) \setminus \{y_2'\}$ and so $N_Y(y_2') \setminus \{y_2\} = N_Y(y_2) \setminus \{y_2'\}$. The inclusion between P_2 and P_1 and between P_2' and P_1 are discussed in the following two cases.

1. $P_2, P_2' \subseteq P_1$. If $\Gamma[P_1]$ is complete, then $\Gamma[P_2]$ and $\Gamma[P_2']$ are complete graphs in which $\Gamma[P_2] \sim \Gamma[P_2']$. So, $y_2y_2' \in E(Y)$ which contradicts by reducibility of Γ over Y. If $\Gamma[P_1]$ is an empty graph, then $\Gamma[P_2]$ and $\Gamma[P_2']$ are empty and so $\Gamma[P_2] \not\sim \Gamma[P_2']$. Thus, $y_2y_2' \notin E(Y)$ which again leaded us to a contradiction with reducibility of Γ over Y.
2. $P_2 \cup P_2' \nsubseteq P_1$. Without loss of generality we assume that $P_2 \nsubseteq P_1$ and $P_1 \cap P_2 \neq \emptyset$. Let $a \in P_1 \cap P_2$, $b \in P_1 \cap P_2'$ and $c \in P_2 \setminus P_1$. There is $P_1' \in \mathcal{P}_1$ such that $c \in P_1'$. It is easy to see that $\Gamma[P_1]$ is complete if and only if $ab \in E(\Gamma)$. Now a similar argument as above shows that $\Gamma[P_2]$ is a complete graph if and only if $ac \in E(\Gamma)$. Hence,

$$\begin{aligned} ab \in E(\Gamma) &\Leftrightarrow \Gamma[P_2] \sim \Gamma[P_2'] \\ &\Leftrightarrow bc \in E(\Gamma) \\ &\Leftrightarrow \Gamma[P_1] \sim \Gamma[P_1'] \\ &\Leftrightarrow ac \in E(\Gamma), \end{aligned}$$

which implies that $\Gamma[P_1]$ is complete if and only if $\Gamma[P_2]$ is a complete graph. Furthermore, it is easy to prove that $\Gamma[P_2] \sim \Gamma[P_2']$ if and only if $y_2y_2' \in E(Y)$. If $P_2' \subseteq P_1$, then $\Gamma[P_1]$ is a complete graph if and only if $\Gamma[P_2']$ is complete which contradicts by reducibility of Γ over Y. Moreover, if $P_2' \nsubseteq P_1$ and $P_1 \cap P_2' \neq \emptyset$, then a similar argument as the case of P_2, shows that $\Gamma[P_1]$ is complete if and only if $\Gamma[P_2']$ is a complete graph which is an another contradiction.

Finally, if there exists an element in $\mathcal{P}_2$ containing P_1, then by changing the role of X and Y, we lead to our final contradiction. This completes the proof. □

By previous theorem, if Γ can be written as a reduced complete-empty $X-$join, then we say that X is a characteristic graph of Γ and denote it by $\chi(\Gamma)$. The following corollary is an immediate consequence of the previous theorem.

Corollary 3.4. *Suppose Γ is a connected NDC graph and $Aut(\Gamma)$ is simple. Then $\Gamma \cong \chi(\Gamma)$.*

Proof. Theorems 3.1 and 3.3 conclude that the graph Γ can be written as a reduced complete-empty $X-$join of some graphs for a unique graph X. If we denote X_x as a graph corresponding to the vertex $x \in V(X)$, then By Corollary 2.4, $Aut(X_x)$ is isomorphic to a normal subgroup of $Aut(\Gamma)$. Since $Aut(X_x) \cong Sym(V(X_x))$ and $Aut(\Gamma)$ is simple, $|V(X_x)| = 1$ which shows that $\Gamma = X = \chi(\Gamma)$. □

4. Concluding Remark

In this paper, the reduced complete-empty $X-$join of graphs together with their automorphism group were studied. It is proved that the automorphism group of such graphs can be written as semi-direct product of two groups. Our calculations with graphs of small orders suggest the following open question:

Question 4.1. *Is it true that every simple connected graph can be written as a reduced complete-empty $X-$join of some graphs?*

It is well-known that most of graphs have trivial automorphism group. If the above question has an affirmative answer then we can immediately prove that the most of graphs have trivial X_x, $x \in V(X)$.

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