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H*-CONDITION ON THE SET OF SUBMODULES OF A MODULE

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Abstract. In this work, we introduce H^*-condition on the set of submodules of a module. Let M be a module. We say M satisfies H^* provided that for every submodule N of M, there is a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. We show that over a right perfect right GV-ring, a homomorphic image of a H^* duo module satisfies H^*.

1. Introduction

All rings considered in this work are associative with identity and all modules are unitary right R-modules. Let M and N be R-modules. It is useful to indicate that by $N \leq M$, we mean that N is a submodule of M. A submodule N of M is said to be *small* in M if $N + K \neq M$ for any proper submodule K of M, and we denote it by $N \ll M$. Maybe, the most important concept in module theory which is closely related to smallness, is lifting modules. A module M is called *lifting*, provided every submodule N of M contains a direct summand D of M such that $N/D \ll M/D$. A number of results concerning lifting modules

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have been appeared in the literature in recent years and many generalizations of the concept of lifting modules have been introduced and studied by several researchers.

Recall from [4] that a module M is called *H-supplemented* in case for every submodule N of M, there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every submodule X of M. Different definition's style, unusual properties and being a generalization of lifting modules, all led many researchers to study and investigate H-supplemented modules further than Mohamed and Müller's elementary introduction in [4]. After introducing this new notation, some authors tried to investigate them more. Maybe first serious effort has been done in [3]. Some general properties of H-supplemented modules such as homomorphic images and direct summands of this class of modules were investigated in [3]. After that in [2], the authors presented some equivalent conditions for a module to be H-supplemented that shows that this class of modules is closely related to the concept of small submodules. In fact in [2], the authors proved that a module M is H-supplemented if and only if for every submodule N of M there is a direct summand D of M such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$. Talebi and his coauthors in [7] studied the concept of H-supplemented modules via preradicals. If τ indicates a preradical, they call in [7] a module M is τ-H-supplemented provided for every submodule N of M, there is a direct summand D of M such that $(N + D)/N \subseteq \tau(M/N)$ and $(N + D)/D \subseteq \tau(M/D)$. In fact this definition somehow develops the concept of H-supplemented modules. There are some works that are relevant to this strange class of modules containing some further properties of them and introducing some generalizations of H-supplemented modules ([5, 11, 12, 13]).

In this work it is important to know the definition of dual singular (nonsingular) modules. Let M be a right R-module. Then M is said to be *(non)cosingular* provided ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$, where $\overline{Z}(M) = \bigcap\{Kerf \mid f : M \to U\}$ in which U is an arbitrary small right R-module.

In [9], the authors introduced a new generalization of lifting modules. They call a module M, C^* provided that every submodule N of M contains a direct summand D of M such that N/D is cosingular. In fact, instead of smallness in the definition of lifting modules, the authors used the concept of a coingular module. Rings for which every right module satisfies C^* are also characterized.

Inspiring by works mentioned, in this manuscript we are interested in studying H-supplemented modules without the concept of small submodules. In other words, we remove the smallness and instead use the concept cosingularity. We say that a module M satisfies H^* in case for every submodule N of M there is a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. We try to study some general properties of modules satisfying

H^*. Over a right V-ring, we proved that a module M satisfies H^* if and only if M is semisimple. We provide some conditions to ensure that a homomorphic image of a module satisfying H^* is H^*.

2. Modules satisfying H^*-condition

In what follows, we shall introduce and study a new generalization of H-supplemented modules by removing the smallness in the definition of this class. The key concept in our work is the cosingularity. In [2], the authors proved that a module M is H-supplemented if and only if for every submodule N of M there exists a direct summand D of M such that $(N + D)/D \ll M/D$ and $(N + D)/N \ll M/N$. As a natural generalization, we have the following definition based on equivalent condition stated for H-supplemented modules.

Definition 2.1. We say that a module M satisfies H^* or is a H^*-module, if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N$ and $(N + D)/D$ are cosingular.

It is not hard to verify that a module M satisfies H^* if and only if for every submodule N of M there exist a submodule X of M and a direct summand D of M such that $N, D \subseteq X$ and X/N and X/D are cosingular.

We next present some examples of modules satisfying H^*.

Example 2.2. (1) Every cosingular module is H^*. In particular every small module satisfies H^*. Hence every commutative domain R over itself satisfies H^* (note that it is known that, for a commutative domain R, the module R_R is a small module.)

(2) It is clear by definitions that every H-supplemented module satisfies H^*. In general, the converse does not hold. Consider $M = \mathbb{Z}$ as an $\mathbb{Z}$-module. Then M satisfies H^* by (1) while M is not H-supplemented. In fact M has no nonzero small submodules.

(3) Every module which satisfies C^* will be H^*. Therefore, every right R-module over a right Harada ring satisfies H^* since over a right Harada ring every right R-module can be expressed as a direct sum of an injective right R-module and a small right R-module (see [1, 28.10]).

The following deals with indecomposable modules satisfying C^* and H^*.

Proposition 2.3. *Let M be an indecomposable module. Then*

(1) *M satisfies C^* if and only if every submodule of M is cosingular.*

(2) *M satisfies H^* if and only if for every submodule N of M, either M/N or N is cosingular.*

Proof. (1) It is easy to check.

(2) Let M be an indecomposable module which satisfies H^*. Suppose that N is an arbitrary submodule of M. Then there is a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. As M is indecomposable, $D=0$ or $D=M$. If $D=0$, then N is cosingular. Otherwise, M/N will be cosingular. The converse is straightforward. □

To find a module satisfying H^* which is not C^*, we should construct an indecomposable module M such that M has a submodule N that M/N is cosingular while N is not a cosingular module. For example, consider an indecomposable module M such that $\overline{Z}(M) \neq 0$ and $\overline{Z}(\overline{Z}(M)) \neq 0$.

It is clear by definitions that every C^*-module satisfies H^*. We shall prove that for a noncosingular module, the two concepts C^* and H^* coincide.

Proposition 2.4. *Let M be a noncosingular module. Then M satisfies C^* if and only if M satisfies H^*.*

Proof. Let M satisfies H^*. Take N as an arbitrary submodule of M. Then there is a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. Since M is noncosingular, D is noncosingular. In this direction, $(N+D)/N$ is noncosingular as a homomorphic image of D. It follows that $N+D=N$ which implies that D is contained in N. In fact N/D is a cosingular module. Hence M satisfies C^*. The converse is obvious. □

A ring R is said to be a right V-ring (GV-ring), in case every simple (singular) right R-module is injective. It follows from [10, Proposition 2.5] that R is a right V-ring if and only if every right R-module is noncosingular.

Corollary 2.5. *Let R be a right V-ring and M a right R-module. Then the following statements are equivalent:*

(1) *M satisfies H^*;*

(2) *M satisfies C^*;*

(3) *M is semisimple.*

Proof. (1) ⇒ (2) It follows from Proposition 2.4 and the fact that over a right V-ring, every right module is noncosingular.

(2) ⇒ (3) It is clear by definitions.

(3) ⇒ (1) Straightforward. □

Proposition 2.6. *Let M be a module and L a submodule of M such that for every decomposition $M = M_1 \oplus M_2$ we have $L = (L \cap M_1) \oplus (L \cap M_2)$. If M satisfies H^*, then L satisfies H^*.*

Proof. Let N be an arbitrary submodule of L. Since M is H^*, there exists a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. Let $D \oplus D' = M$. By assumption $L = (L \cap D) \oplus (L \cap D')$. Now $[(L \cap D) + N]/N \subseteq (N + D)/N$ is cosingular. Consider the submodule $[(L \cap D) + N]/(L \cap D) \cong N/(N \cap L \cap D) = N/(N \cap D) \cong (N + D)/D$. Therefore $[(L \cap D) + N]/(L \cap D)$ is cosingular. Note that $L \cap D$ is a direct summand of L. Consequently L satisfies H^*. □

Recall that a submodule N of a module M is (projection invariant) fully invariant in case for every (idempotent) endomorphism f of M, we have $f(N) \subseteq N$.

Corollary 2.7. *Every projection invariant (fully invariant) submodule of a H^* module satisfies H^*.*

Recall that a module M has *Summand Intersection Property* (SIP), in case intersection of each pair of direct summands of M is again a direct summand of M.

Proposition 2.8. *Let M be a H^*-module with (SIP). Then every direct summand of M is H^*.*

Proof. The proof is the same of the proof of Proposition 2.6. □

We shall investigate homomorphic images of H^*-modules. Note that a homomorphic image of a cosingular module need not be cosingular. For example, $\mathbb{Q}$ as an $\mathbb{Z}$-module which is a homomorphic image of the cosingular $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$, is noncosingular (see [10, Remark 2.11(1)]). Before dealing with homomorphic images of H^*-modules, we present some conditions under which any homomorphic image of a cosingular module is cosingular. The following is a consequence of [8, Theorem 3.1] and [8, Proposition 3.5].

Proposition 2.9. *Let R be a ring such that every simple cosingular R-module is projective and each cosingular module is amply supplemented. Then, the class of cosingular R-modules is closed under taking homomorphic images.*

Proof. Let M be a cosingular R-module and $0 \neq x \in M$. Then xR has a maximal submodule K. So xR/K is simple. If xR/K is injective, then it is non-cosingular. Consider natural epimorphism $\pi : xR \to xR/K$. By assumption xR, is amply supplemented. Now by [10, Corollary 3.6], $0 = \pi(\overline{Z}^2(xR)) = \overline{Z}^2(xR/K) = \overline{Z}(xR/K) = xR/K$, a contradiction. Hence,

xR/K is small and since every simple cosingular R-module is projective, xR/K is projective. So K is a direct summand of xR. Therefore, M is semisimple. Then every homomorphic image of M, is isomorphic to a submodule of M. This completes the proof. □

We shall introduce a ring satisfying the conditions in Proposition 2.9.

Example 2.10. Let F be a field and let R be the ring of all upper triangular 2×2 matrices with entries from F. Then by [8, Example 3.15], R is a left and right perfect GV-ring. Hence every homomorphic image of a cosingular right R-module is cosingular. Note that by [8, Example 3.15], every cosingular module is projective. So, if M is a right R-module satisfying H^*, N is an arbitrary submodule of M and D its associated direct summand (according to the definition of H^*-modules), then N will be a direct summand of $N + D$.

By [6, Example 4.16] a homomorphic image of a module satisfying H^* does not necessarily H^*. In fact there is an injective $\mathbb{Z}$-module which is a homomorphic image of $\mathbb{Z}^{(I)}$ and does not satisfy H^*.

Proposition 2.11. *Let R be a ring such that the class of cosingular right R-modules is closed under homomorphic images (see Proposition 2.9). Let M be a H^*-module and $X \leq M$ such that for every direct summand D of M, $(X+D)/X$ is a direct summand of M/X. Then M/X is H^*.*

Proof. Let $N/X \leq M/X$. Then for submodule N of M there exists a direct summand D of M such that $(N+D)/N$ and $(N+D)/D$ are cosingular. Now, $\frac{N/X+(D+X)/X}{N/X} = \frac{(N+D)/X}{N/X} \cong (N+D)/N$ is cosingular. Now consider $\frac{N/X+(D+X)/X}{(D+X)/X} = \frac{(N+D)/X}{(X+D)/X} \cong (N+D)/(X+D)$. Next, $(N+D)/(X+D) \cong \frac{(N+D)/D}{(X+D)/D}$. Since $(N+D)/D$ is cosingular, by assumption $(N+D)/(X+D)$ is cosingular. Because $(X+D)/X$ is a direct summand of M/X, it follows that M/X is H^*. □

Corollary 2.12. *Let R be a right perfect right GV-ring. If M is a right R-module satisfying H^* and L a fully invariant submodule of M, then L and M/L satisfy H^*.*

Proof. First of all we shall note that since L is a fully invariant submodule of M, then for every direct summand D of M, the submodule $(D+L)/L$ is a direct summand of M/L. By [8, Theorem 3.1] and Proposition 2.9, every homomorphic image of a cosingular right R-module is cosingular. Now, M/L will satisfy H^* by Proposition 2.11. The rest is followed by Proposition 2.6. □

It follows from [6, Remark 4.20], the class of H^*-modules is not closed under direct sums.

Theorem 2.13. *Let $M = M_1 \oplus M_2$ be a duo module. Then M_1 and M_2 are H^* if and only if M is H^*.*

Proof. Let $N \leq M$. Since M is a duo module, $N = (N \cap M_1) \oplus (N \cap M_2)$. Now for submodules $N \cap M_1$ of M_1 and $N \cap M_2$ of M_2, since M_1 and M_2 are H^*, there exist direct summands D_1 of M_1 and D_2 of M_2 such that $[(N \cap M_i) + D_i]/D_i$, $[(N \cap M_i) + D_i]/(N \cap M_i)$ are cosingular for $i \in \{1, 2\}$. Set $D = D_1 \oplus D_2$, which is a direct summand of M. Now $(N + D)/D = ([(N \cap M_1) \oplus (N \cap M_2)] + [D_1 \oplus D_2])/(D_1 \oplus D_2) \cong [(N \cap M_1) + D_1]/D_1 \oplus [(N \cap M_2) + D_2]/D_2$ is cosingular by [10, Corollary 2.2]. Also $(N + D)/N = [(N \cap M_1) \oplus (N \cap M_2)] + [D_1 \oplus D_2]/[(N \cap M_1) \oplus (N \cap M_2)] \cong [(N \cap M_1) + D_1]/(N \cap M_1) \oplus [(N \cap M_2) + D_2]/(N \cap M_2)$ is cosingular by same argument. It follows that M is H^*. The converse follows directly from Proposition 2.6. □

Proposition 2.14. *Let M be a module such that for every $N \leq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is cosingular. If M is projective, then M satisfies H^*.*

Proof. Let $N \leq M$. By assumption, there exists a direct summand M_2 of M such that $M = N + M_2$ and $N \cap M_2$ is cosingular. Let $M = M_1 \oplus M_2$. Since M_1 is M_2-projective, there exists a submodule A of N such that $M = A \oplus M_2$. Then by modular low, $N = A \oplus (N \cap M_2)$. It is clear that $(N + A)/A$ and $(N + A)/N$ are cosingular. Hence M satisfies H^*. □

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