Abstract. Let $A$ be a commutative ring with nonzero identity, and $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). The total dot product graph of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0,0,\ldots,0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$). Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0,0,\ldots,0)\}$. It follows that if $\Gamma(A)$ is not perfect, then $ZD(R)$ (and hence $TD(R)$) is not perfect. In this paper we investigate perfectness of the graphs $TD(R)$ and $ZD(R)$.

1. Introduction

Let $R$ be a commutative ring with nonzero identity. Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph-theoretic language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see

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Probably the most attention has been to the zero-divisor graph $\Gamma(R)$ for a commutative ring $R$. Let $Z(R)$ be the set of all zero-divisor elements of $R$. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$.

We use the standard terminology of graphs following [8]. Let $G = (V,E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of $G$. We write $u - v$, to denote an edge with ends $u, v$. A graph $H = (V_0, E_0)$ is called a subgraph of $G$ if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, $H$ is called an induced subgraph by $V_0$, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{(u, v) \in E \mid u, v \in V_0\}$. The graph obtained by taking the union of graphs $G_1$ and $G_2$ with disjoint vertex sets is the disjoint union or sum, written $G_1 + G_2$. For a graph $G$, $S \subseteq V(G)$ is called a clique if the subgraph induced on $S$ is complete. The number of vertices in the largest clique of graph $G$ is called the clique number of $G$ and is often denoted by $\omega(G)$. For a graph $G$, let $\chi(G)$ denote the chromatic number of $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Clearly, for every graph $G$, $\omega(G) \leq \chi(G)$. A graph $G$ is said to be weakly perfect if $\omega(G) = \chi(G)$. A perfect graph $G$ is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Let $A$ be a commutative ring with nonzero identity, and $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). The total dot product graph of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0,0,\ldots,0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$ i.e., if $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$, then $x \cdot y = \Sigma_{i=1}^n x_i y_i$ (see [9])). Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0,0,\ldots,0)\}$. In this paper we investigate perfectness of the graphs $TD(R)$ and $ZD(R)$.

2. Some perfect total dot product graphs and zero-divisor dot product graphs of rings

In this section at first we give an example of $R$ to show that $TD(R)$ is not a perfect graph in general and also we give another example of $R$ that shows $ZD(R)$ is perfect graph.

Example 2.1.

(1) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $A = \{(1,0,1),(0,1,0),(1,0,0),(0,1,1),(1,1,1)\}$, $B = \{(1,0,1),(0,1,0),(0,0,1),(1,1,0),(1,1,1)\}$.
In the two induced subgraphs of $TD(R)$ on $A, B$, both of these graphs have the size of the largest cliques equal to 2 but the chromatic numbers of them is 3. This clearly concludes that $TD(R)$ is not a perfect graph.

(2) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following figure shows that for every induced subgraph of $ZD(R)$, its clique number is equal to its chromatic number and thus $ZD(R)$ is a perfect graph.

Let $G$ be a graph. Since the number of subgraphs in the graph $G$ may be large enough, by definition, it is not practically possible to obtain that $G$ is perfect. So we introduce the following theorem, which is another way to do this.

**Theorem 2.2.** [3, The Strong Perfect Graph Theorem] A graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ contains an induced odd cycle of length at least 5.

**Lemma 2.3.** Let $2 \leq n < \infty$, $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times \cdots \times A$ ($n$ times). If $n \geq 4$, then $ZD(R)$ (and hence $TD(R)$) is not perfect.
Then the following statements hold:

\[ A \times \text{no induced odd cycle of length at least 5 and so we have two claims:} \]

For this by Theorem 2.4. Hence \[ \text{TD} \]

\[ Z \text{TD} \]

Theorem 2.4. Let \( 2 \leq n < \infty \), \( A \) be an integral domain and \( R = A \times A \times \cdots \times A \) (\( n \) times).

Then the following statements hold:

1. \( n = 2 \) if and only if \( \text{TD}(R) \) is perfect.
2. \( n \leq 3 \) if and only if \( \text{ZD}(R) \) is perfect.

Proof. (1) Suppose that \( \text{TD}(R) \) is perfect. If \( n \neq 2 \), then by Lemma 2.3, \( n = 3 \). So \( R = A \times A \times A \). Now, it is easy to check that \( a - b - c - d - e - a \) is an induced odd cycle of length 5 in \( \text{TD}(R) \), where

\[ a = (1, 1, -1), b = (1, 0, 1), c = (0, 1, 0), d = (1, 0, 0), e = (0, 1, 1). \]

So \( \text{TD}(R) \) (and hence \( \text{TD}(R) \)) is not perfect by Theorem 2.2. Hence \( \text{TD}(R) \) is an induced odd cycle of length at least 5.

Conversely, suppose that \( n = 2 \). So \( R = A \times A \). We show that \( \text{TD}(R) \) is perfect. Let \( \overline{Z(R)} = R \setminus Z(R) \). We can easily get \( \text{TD}(R)[\overline{Z(R)}] \) and \( \text{ZD}(R) \) are disjoint graphs and hence \( \text{TD}(R) = \text{TD}(R)[\overline{Z(R)}] + \text{ZD}(R) \). On the other hand, by [2, Theorem 2.1], we have \( \text{ZD}(R) = \Gamma(R) \). Thus we need to show that \( \Gamma(R) \) is perfect. Since \( R = A \times A \), \( \text{V}(\Gamma(R)) = Z(R)^* = \{(a, 0), (0, a) \mid a \in A^*\} \). If we put \( X = \{(a, 0) \mid a \in A^*\} \) and \( Y = \{(0, a) \mid a \in A^*\} \), then we can easily get \( \Gamma(R) = K_{|X|,|Y|} \). This implies that \( \Gamma(R) \) is a complete bipartite graph and hence is perfect. So we need only to show that \( \text{TD}(R)[\overline{Z(R)}] \) is perfect.

For this by Theorem 2.2, it is enough to show that \( \text{TD}(R)[\overline{Z(R)}] \) and \( \text{TD}(R)[\overline{Z(R)}] \) contain no induced odd cycle of length at least 5 and so we have two claims:

Claim 1. \( \text{TD}(R)[\overline{Z(R)}] \) contains no induced odd cycle of length at least 5. Let

\[ x_1 - x_2 - \cdots - x_n - x_1 \]

is an induced odd cycle of length at least 5 in \( \text{TD}(R)[\overline{Z(R)}] \). We show that \( x_1 \) and \( x_4 \) are adjacent. For some \( a, b, c, d, a', b', c', d' \in A \), let

\[ x_1 = (a, a'), x_2 = (b, b'), x_3 = (c, c'), x_4 = (d, d'). \]

Since \( x_1 \) and \( x_2 \) are adjacent, we have \( ab = -a'b' \). Similarly, \( bc = -b'c' \) and \( cd = -c'd' \).

Since \( ab = -a'b' \) and \( cd = -c'd' \), \( abcd = a'b'c'd' \). Since \( bc = -b'c' \), we have \( ad = -a'd' \). This, clearly follows that \( x_1 \) and \( x_4 \) are adjacent, a contradiction. Hence \( \text{TD}(R)[\overline{Z(R)}] \) contains no induced odd cycle of length at least 5.
Claim 2. \( TD(R)[Z(R)] \) contains no induced odd cycle of length at least 5. The argument is similar to the proof of Claim 1. Let 

\[
x_1 - x_2 - \cdots - x_n - x_1
\]

is an induced odd cycle of length at least 5 in \( TD(R)[Z(R)] \). Let

\[
x_1 = (a, a'), x_2 = (b, b'), x_3 = (c, c'), x_4 = (d, d').
\]

we can easily get \( ac = -a'c', bd = -b'd' \). So we have \( acbd = a'c'b'd' \) and hence since \( ad = -a'd', cb = -c'b' \), a contradiction. Hence \( TD(R)[Z(R)] \) contains no induced odd cycle of length at least 5. By Claim 1, 2 and Theorem 2.2, we have \( TD(R) \) is a perfect graph.

(2) If \( ZD(R) \) is perfect, then clearly by Lemma 2.3, \( n \leq 3 \).

Conversely, suppose that \( n \leq 3 \). We show that \( ZD(R) \) is perfect. If \( n = 2 \), then by part (1) \( ZD(R) \) is perfect. So we let \( n = 3 \) and show that \( ZD(R) \) is perfect. For this case by Theorem 2.2, it is enough to show that \( ZD(R) \) and \( ZD(R) \) contain no induced odd cycle of length at least 5 and so we have two claims:

Claim 1. \( ZD(R) \) contains no induced odd cycle of length at least 5. Let

\[
x_1 - x_2 - \cdots - x_n - x_1
\]

is an induced odd cycle of length at least 5 in \( ZD(R) \). For every \( 1 \leq i \leq n \), let \( x_i = (a_{i1}, b_{i2}, c_{i3}) \).

Since for every \( 1 \leq i \leq n \), \( x_i \in Z(R) \), with no loss of generality, we may assume that \( x_1 = (0, b_{12}, c_{13}) \). We show that \( b_{12} \neq 0 \) and \( c_{13} \neq 0 \). If \( b_{12} = 0 \), then \( x_1 = (0, 0, c_{13}) \) and since \( x_1 \cdot x_2 = 0 \), \( x_1 \cdot x_3 \neq 0 \) and \( x_1 \cdot x_4 \neq 0 \), it is easy to check that \( c_{23} = 0, c_{33} \neq 0 \) and \( c_{43} \neq 0 \). On the other hand, since \( x_3 \cdot x_4 = 0, c_{33} \neq 0 \) and \( c_{43} \neq 0 \), we have \( a_{31} = a_{41} = 0 \) or \( b_{32} = b_{42} = 0 \). We can let \( a_{31} = a_{41} = 0 \) and hence \( b_{32} \neq 0 \) and \( b_{42} \neq 0 \). Since \( x_2 \cdot x_3 = 0 \), we conclude that \( x_2 = (a_{21}, 0, 0) \) and so \( x_2 \cdot x_4 = 0 \), a contradiction. Hence \( b_{12} \neq 0 \). Similarly, \( c_{13} \neq 0 \) and thus \( x_1 = (0, b_{12}, c_{13}) \) such that \( b_{12} \neq 0 \) and \( c_{13} \neq 0 \). Again, this implies that for every \( 1 \leq i \leq n \), only one component of \( x_i \) is 0. This form of \( x_1 \), together with \( x_1 \cdot x_2 = 0 \) implies that \( a_{21} = 0, b_{22} \neq 0 \) and \( c_{23} \neq 0 \). We continue this procedure for \( n \) times, and obtain that for every \( 1 \leq i \leq n \), \( a_{i1} = 0, b_{i2} \neq 0 \) and \( c_{i3} \neq 0 \). This clearly follows that

\[
x'_1 - x'_2 - \cdots - x'_n - x'_1
\]

is an induced odd cycle of length at least 5 in \( TD(S) \), where \( S = A \times A \) and for every \( 1 \leq i \leq n \), \( x'_i = (b_{i2}, c_{i3}) \). But, this is a contradiction, as \( TD(S) \) contains no induced odd cycle of length at least 5 by part (1) and Theorem 2.2. Hence \( ZD(R) \) contains no induced odd cycle of length at least 5.
Claim 2. $\overline{ZD}(R)$ contains no induced odd cycle of length at least 5. The argument is similar to the proof of Claim 1. Let

$$x_1 - x_2 - \cdots - x_n - x_1$$

is an induced odd cycle of length at least 5 in $\overline{ZD}(R)$. For every $1 \leq i \leq n$, let $x_i = (a_i, b_i, c_i)$.

We may assume that $x_1 = (0, b_{12}, c_{13})$. We show that $b_{12} \neq 0$ and $c_{13} \neq 0$. If $b_{12} = 0$, then $x_1 = (0, 0, c_{13})$ and thus we can easily get $c_{23} \neq 0$ and $c_{33} = c_{43} = 0$. Now, since $x_2 \cdot x_3 \neq 0$, $b_{22}b_{32} \neq 0$ or $a_{21}a_{31} \neq 0$. We may assume that $b_{22}b_{32} \neq 0$. This implies that

$$b_{22} \neq 0 \text{ and } b_{32} \neq 0, \quad a_{21} = 0.$$  

So we have $x_1 = (0, 0, c_{13}), x_2 = (0, b_{22}, c_{23}), x_3 = (a_{31}, b_{32}, 0), x_4 = (a_{41}, b_{42}, 0), x_5 = (a_{51}, b_{52}, c_{53})$.

Since $x_2 \cdot x_4 = 0$, $b_{42} = 0$. This implies that $a_{51} \neq 0$ and $a_{31} \neq 0$. Now, it is easy to check that $x_5 \cdot x_3 \neq 0$ or $x_5 \cdot x_2 \neq 0$, a contradiction. Hence $b_{12} \neq 0$. Similarly, $c_{13} \neq 0$ and thus $x_1 = (0, b_{12}, c_{13})$ such that $b_{12} \neq 0$ and $c_{13} \neq 0$. Again, this implies that for every $1 \leq i \leq n$, only one component of $x_i$ is 0. Next, we show that $x_2 = (0, b_{22}, c_{23})$. Since $x_1 = (0, b_{12}, c_{13})$ and $x_1 \cdot x_4 = 0$, the form of $x_4$ must be the form $x_4 = (0, b_{12}, c_{13})$. This form of $x_4$, together with $x_4 \cdot x_2 = 0$, implies that $x_2 = (0, b_{22}, c_{23})$. Similarly, for every $1 \leq i \leq n$, $a_{i1} = 0, b_{i2} \neq 0$ and $c_{i3} \neq 0$. Now, by similar proof of Claim 1, we can easily get $\overline{ZD}(R)$ contains no induced odd cycle of length at least 5. By Claim 1, 2 and Theorem 2.2, we have $ZD(R)$ is a perfect graph. \[\square\]

Theorem 2.5. Let $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times A$. Then $ZD(R)$ is perfect if and only if $A$ is an integral domain.

Proof. If $A$ is an integral domain, then by Theorem 2.4, $ZD(R)$ is perfect.

Conversely, suppose that $ZD(R)$ is perfect. If $xy = 0$ for some $x, y \in Z(A)^*$, then it is easy to check that $a - b - c - d - e - a$ is an induced odd cycle of length 5 in $ZD(R)$, where $a = (0, x, 1), b = (1, 0, 0), c = (0, 1, 0), d = (x, 0, 1), e = (y, y, 0)$.

So $ZD(R)$ is not perfect by Theorem 2.4, a contradiction. Hence $A$ is an integral domain. \[\square\]

Corollary 2.6. Let $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times A$. Then $ZD(R)$ is perfect if and only if $\Gamma(R)$ is perfect.

Proof. If $ZD(R)$ is perfect, then by Theorem 2.5, $A$ is an integral domain and thus we can easily get $\Gamma(R)$ is perfect. Conversely, suppose that $\Gamma(R)$ is perfect. By Theorem 2.5, we need only show that $A$ is an integral domain. But if $A$ is not an integral domain, then
\[(0, x, 1) - (1, 0, 0) - (0, 1, 0) - (x, 0, 1) - (y, y, 0) - (0, x, 1)\] is an induced odd cycle of length 5 in \(\Gamma(R)\), where \(xy = 0\) for some \(x, y \in Z(A)^*\), a contradiction. \(\square\)

**Theorem 2.7.** Let \(A\) be a commutative ring with \(1 \neq 0\) and \(R = A \times A\). If \(\Gamma(A)\) contains a path on four vertices, then \(ZD(R)\) is not perfect.

**Proof.** Assume that \(a - b - c - d\) is a path on four vertices. It is easy to check that \(x_1 - x_2 - x_3 - x_4 - x_5 - x_1\) is an induced odd cycle of length 5 in \(ZD(R)\), where \(x_1 = (a, 0), x_2 = (b, -c), x_3 = (c, b), x_4 = (d, 0), x_5 = (0, 1)\).

So \(ZD(R)\) is not perfect by Theorem 2.2 \(\square\)

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