Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 6 No. 2 ( 2019 ) pp 1-7.

# ON PERFECTNESS OF DOT PRODUCT GRAPH OF A COMMUTATIVE RING

NAZI. ABACHI AND SHERVIN. SAHEBI*

ABSTRACT. Let $A$ be a commutative ring with nonzero identity, and $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). The total dot product graph of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$). Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. It follows that if $\Gamma(A)$ is not perfect, then $ZD(R)$ (and hence $TD(R)$) is not perfect. In this paper we investigate perfectness of the graphs $TD(R)$ and $ZD(R)$.

## 1. Introduction

Let $R$ be a commutative ring with nonzero identity. Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph-theoretic language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers

DOI: 10.29252/as.2019.1399

MSC(2010):13A15; 13B99; 05C99; 05C25

Keywords: annihilator graph, Zero-divisor, Complete graph.

Received: 25 Dec 2018, Accepted: 13 May 2019.

*Corresponding author

which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [2, 4, 1, 5] and [3]. Probably the most attention has been to the *zero-divisor graph* $\Gamma(R)$ for a commutative ring $R$. Let $Z(R)$ be the set of all zero-divisor elements of $R$. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$.

We use the standard terminology of graphs following [7]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of $G$. We write $u - v$, to denote an edge with ends $u, v$. A graph $H = (V_0, E_0)$ is called a *subgraph of* $G$ if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, $H$ is called an *induced subgraph by* $V_0$, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. The graph obtained by taking the union of graphs $G_1$ and $G_2$ with disjoint vertex sets is the *disjoint union* or *sum*, written $G_1 + G_2$. For a graph $G$, $S \subseteq V(G)$ is called a *clique* if the subgraph induced on $S$ is complete. The number of vertices in the largest clique of graph $G$ is called the *clique number* of $G$ and is often denoted by $\omega(G)$. For a graph $G$, let $\chi(G)$ denote the *chromatic number* of $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. Clearly, for every graph $G$, $\omega(G) \leq \chi(G)$. A graph $G$ is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* $G$ is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Let $A$ be a commutative ring with nonzero identity, and $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). The total dot product graph of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$ i.e., if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then $x \cdot y = \Sigma_{i=1}^{n} x_i y_i$ (see [5])). Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. In this paper we investigate perfectness of the graphs $TD(R)$ and $ZD(R)$.

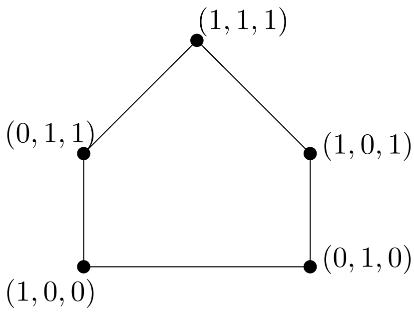
## 2. Some perfect total dot product graphs and zero-divisor dot product graphs of rings

In this section at first we give an example of $R$ to show that $TD(R)$ is not a perfect graph in general and also we give another example of $R$ that shows $ZD(R)$ is perfect graph.

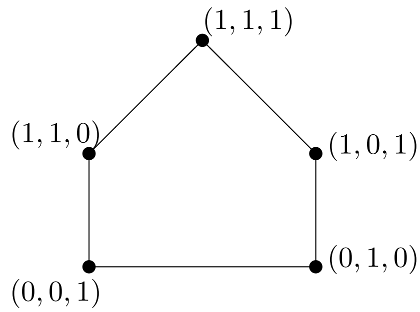
**Example 2.1.**

(1) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $A = \{(1, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 1, 1)\}$,
$B = \{(1, 0, 1), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$.

In the two induced subgraphs of $TD(R)$ on $A, B$, both of these graphs have the size of the largest cliques equal to 2 but the chromatic numbers of them is 3. This clearly concludes that $TD(R)$ is not a perfect graph.

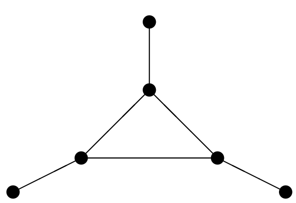



$TD(R)[A]$




$TD(R)[B]$

(2) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following figure shows that for every induced subgraph of $ZD(R)$, its clique number is equal to its chromatic number and thus $ZD(R)$ is a perfect graph.



$ZD(R)$

Let $G$ be a graph. Since the number of subgraphs in the graph $G$ may be large enough, by definition, it is not practically possible to obtain that $G$ is perfect. So we introduce the following theorem, which is another way to do this.

**Theorem 2.2.** [6, The Strong Perfect Graph Theorem] *A graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

**Lemma 2.3.** *Let $2 \leq n < \infty$, $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times \cdots \times A$ ($n$ times). If $n \geq 4$, then $ZD(R)$ ( and hence $TD(R)$) is not perfect.*

*Proof.* Assume that $n \geq 4$. It is easy to check that $a - b - c - d - e - a$ is an induced odd cycle of length 5 in $ZD(R)$, where $a = (1, 1, 0, 0, 0, \dots, 0), b = (1, -1, 1, 0, 0, \dots, 0), c = (0, 1, 1, 1, 0, \dots, 0), d = (0, 1, 0, -1, 0, \dots, 0), e = (0, 0, 1, 0, 0, \dots, 0)$.

So $ZD(R)$ ( and hence $TD(R)$) is not perfect by Theorem 2.2. □

**Theorem 2.4.** *Let $2 \leq n < \infty$, $A$ be an integral domain and $R = A \times A \times \cdots \times A$ ($n$ times). Then the following statements hold:*

(1) *$n = 2$ if and only if $TD(R)$ is perfect.*

(2) *$n \leq 3$ if and only if $ZD(R)$ is perfect.*

*Proof.* (1) Suppose that $TD(R)$ is perfect. If $n \neq 2$, then by Lemma 2.3, $n = 3$. So $R = A \times A \times A$. Now, it is easy to check that $a - b - c - d - e - a$ is an induced odd cycle of length 5 in $TD(R)$, where

$a = (1, 1, -1), b = (1, 0, 1), c = (0, 1, 0), d = (1, 0, 0), e = (0, 1, 1)$.

So $TD(R)$ is not perfect by Theorem 2.2, a contradiction. Hence $n = 2$.

Conversely, suppose that $n = 2$. So $R = A \times A$. We show that $TD(R)$ is perfect. Let $\overline{Z(R)} = R \setminus Z(R)$. We can easily get $TD(R)[\overline{Z(R)}]$ and $ZD(R)$ are disjoint graphs and hence $TD(R) = TD(R)[\overline{Z(R)}] + ZD(R)$. On the other hand, by [5, Theorem 2.1], we have $ZD(R) = \Gamma(R)$. Thus we need to show that $\Gamma(R)$ is perfect. Since $R = A \times A$, $V(\Gamma(R)) = Z(R)^* = \{(a, 0), (0, a) \mid a \in A^*\}$. If we put $X = \{(a, 0) \mid a \in A^*\}$ and $Y = \{(0, a) \mid a \in A^*\}$, then we can easily get $\Gamma(R) = K_{|X|,|Y|}$. This implies that $\Gamma(R)$ $(= ZD(R))$ is a complete bipartite graph and hence is perfect. So we need only to show that $TD(R)[\overline{Z(R)}]$ is perfect. For this by Theorem 2.2, it is enough to show that $TD(R)[\overline{Z(R)}]$ and $\overline{TD(R)[\overline{Z(R)}]}$ contain no induced odd cycle of length at least 5 and so we have two claims:

**Claim 1.** $TD(R)[\overline{Z(R)}]$ contains no induced odd cycle of length at least 5. Let

$$x_1 - x_2 - \cdots - x_n - x_1$$

is an induced odd cycle of length at least 5 in $TD(R)[\overline{Z(R)}]$. We show that $x_1$ and $x_4$ are adjacent. For some $a, b, c, d, a', b', c', d' \in A$, let

$$x_1 = (a, a'), x_2 = (b, b'), x_3 = (c, c'), x_4 = (d, d').$$

Since $x_1$ and $x_2$ are adjacent, we have $ab = -a'b'$. Similarly, $bc = -b'c'$ and $cd = -c'd'$.

Since $ab = -a'b'$ and $cd = -c'd'$, $abcd = a'b'c'd'$. Since $bc = -b'c'$, we have $ad = -a'd'$. This, clearly follows that $x_1$ and $x_4$ are adjacent, a contradiction. Hence $TD(R)[\overline{Z(R)}]$ contains no induced odd cycle of length at least 5.

**Claim 2.** $\overline{TD(R)[\overline{Z(R)}]}$ contains no induced odd cycle of length at least 5. The argument is similar to the proof of Claim 1. Let

$$x_1 - x_2 - \cdots - x_n - x_1$$

is an induced odd cycle of length at least 5 in $\overline{TD(R)[\overline{Z(R)}]}$. let

$$x_1 = (a, a'), x_2 = (b, b'), x_3 = (c, c'), x_4 = (d, d').$$

we can easily get $ac = -a'c'$, $bd = -b'd'$. So we have $acbd = a'c'b'd'$ and hence since $ad = -a'd'$, $cb = -c'b'$, a contradiction. Hence $\overline{TD(R)[\overline{Z(R)}]}$ contains no induced odd cycle of length at least 5. By Claim 1, 2 and Theorem 2.2, we have $TD(R)$ is a perfect graph.

(2) If $ZD(R)$ is perfect, then clearly by Lemma 2.3, $n \leq 3$.

Conversely, suppose that $n \leq 3$. We show that $ZD(R)$ is perfect. If $n = 2$, then by part (1) $ZD(R)$ is perfect. So we let $n = 3$ and show that $ZD(R)$ is perfect. For this case by Theorem 2.2, it is enough to show that $ZD(R)$ and $\overline{ZD(R)}$ contain no induced odd cycle of length at least 5 and so we have two claims:

**Claim 1.** $ZD(R)$ contains no induced odd cycle of length at least 5. Let

$$x_1 - x_2 - \cdots - x_n - x_1$$

is an induced odd cycle of length at least 5 in $ZD(R)$. For every $1 \leq i \leq n$, let $x_i = (a_{i1}, b_{i2}, c_{i3})$.

Since for every $1 \leq i \leq n$, $x_i \in Z(R)$, with no loss of generality, we may assume that $x_1 = (0, b_{12}, c_{13})$. We show that $b_{12} \neq 0$ and $c_{13} \neq 0$. If $b_{12} = 0$, then $x_1 = (0, 0, c_{13})$ and since $x_1 \cdot x_2 = 0$, $x_1 \cdot x_3 \neq 0$ and $x_1 \cdot x_4 \neq 0$, it is easy to check that $c_{23} = 0$, $c_{33} \neq 0$ and $c_{43} \neq 0$. On the other hand, since $x_3 \cdot x_4 = 0$, $c_{33} \neq 0$ and $c_{43} \neq 0$, we have $a_{31} = a_{41} = 0$ or $b_{32} = b_{42} = 0$. We can let $a_{31} = a_{41} = 0$ and hence $b_{32} \neq 0$ and $b_{42} \neq 0$ . Since $x_2 \cdot x_3 = 0$, we conclude that $x_2 = (a_{21}, 0, 0)$ and so $x_2 \cdot x_4 = 0$, a contradiction. Hence $b_{12} \neq 0$. Similarly, $c_{13} \neq 0$ and thus $x_1 = (0, b_{12}, c_{13})$ such that $b_{12} \neq 0$ and $c_{13} \neq 0$. Again, this implies that for every $1 \leq i \leq n$, only one component of $x_i$ is 0. This form of $x_1$, together with $x_1 \cdot x_2 = 0$ implies that $a_{21} = 0$, $b_{22} \neq 0$ and $c_{23} \neq 0$. We continue this procedure for $n$ times, and obtain that for every $1 \leq i \leq n$, $a_{i1} = 0$, $b_{i2} \neq 0$ and $c_{13} \neq 0$. This clearly follows that

$$x'_1 - x'_2 - \cdots - x'_n - x'_1$$

is an induced odd cycle of length at least 5 in $TD(S)$, where $S = A \times A$ and for every $1 \leq i \leq n$, $x'_i = (b_{i2}, c_{i3})$. But, this is a contradiction, as $TD(S)$ contains no induced odd cycle of length at least 5 by part (1) and Theorem 2.2. Hence $ZD(R)$ contains no induced odd cycle of length at least 5.

**Claim 2.** $\overline{ZD(R)}$ contains no induced odd cycle of length at least 5. The argument is similar to the proof of Claim 1. Let

$$x_1 - x_2 - \cdots - x_n - x_1$$

is an induced odd cycle of length at least 5 in $\overline{ZD(R)}$. For every $1 \leq i \leq n$, let $x_i = (a_{i1}, b_{i2}, c_{i3})$.

We may assume that $x_1 = (0, b_{12}, c_{13})$. We show that $b_{12} \neq 0$ and $c_{13} \neq 0$. If $b_{12} = 0$, then $x_1 = (0, 0, c_{13})$ and thus we can easily get $c_{23} \neq 0$ and $c_{33} = c_{43} = 0$. Now, since $x_2 \cdot x_3 \neq 0$, $b_{22}b_{32} \neq 0$ or $a_{21}a_{31} \neq 0$. We may assume that $b_{22}b_{32} \neq 0$. This implies that

$b_{22} \neq 0$ and $b_{32} \neq 0$, $a_{21} = 0$. So we have $x_1 = (0, 0, c_{13}), x_2 = (0, b_{22}, c_{23}), x_3 = (a_{31}, b_{32}, 0), x_4 = (a_{41}, b_{42}, 0), x_5 = (a_{51}, b_{52}, c_{53})$.

Since $x_2 \cdot x_4 = 0$, $b_{42} = 0$. This implies that $a_{51} \neq 0$ and $a_{31} \neq 0$. Now, it is easy to check that $x_5 \cdot x_3 \neq 0$ or $x_5 \cdot x_2 \neq 0$, a contradiction. Hence $b_{12} \neq 0$. Similarly, $c_{13} \neq 0$ and thus $x_1 = (0, b_{12}, c_{13})$ such that $b_{12} \neq 0$ and $c_{13} \neq 0$. Again, this implies that for every $1 \leq i \leq n$, only one component of $x_i$ is 0. Next, we show that $x_2 = (0, b_{22}, c_{23})$. Since $x_1 = (0, b_{12}, c_{13})$ and $x_1 \cdot x_4 = 0$, the form of $x_4$ must be the form $x_4 = (0, b_{42}, c_{43})$. This form of $x_4$, together with $x_4 \cdot x_2 = 0$, implies that $x_2 = (0, b_{22}, c_{23})$. Similarly, for every $1 \leq i \leq n$, $a_{i1} = 0$, $b_{i2} \neq 0$ and $c_{i3} \neq 0$. Now, by similar proof of Claim 1, we can easily get $\overline{ZD(R)}$ contains no induced odd cycle of length at least 5. By Claim 1, 2 and Theorem 2.2, we have $ZD(R)$ is a perfect graph. □

**Theorem 2.5.** *Let $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times A$. Then $ZD(R)$ is perfect if and only if $A$ is an integral domain.*

*Proof.* If $A$ is an integral domain, then by Theorem 2.4, $ZD(R)$ is perfect.

Conversely, suppose that $ZD(R)$ is perfect. If $xy = 0$ for some $x, y \in Z(A)^*$, then it is easy to check that $a - b - c - d - e - a$ is an induced odd cycle of length 5 in $ZD(R)$, where $a = (0, x, 1), b = (1, 0, 0), c = (0, 1, 0), d = (x, 0, 1), e = (y, y, 0)$.

So $ZD(R)$ is not perfect by Theorem 2.2, a contradiction. Hence $A$ is an integral domain.

□

**Corollary 2.6.** *Let $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A \times A$. Then $ZD(R)$ is perfect if and only if $\Gamma(R)$ is perfect.*

*Proof.* If $ZD(R)$ is perfect, then by Theorem 2.5, $A$ is an integral domain and thus we can easily get $\Gamma(R)$ is perfect. Conversely, suppose that $\Gamma(R)$ is perfect. By Theorem 2.5, we need only show that $A$ is an integral domain. But if $A$ is not an integral domain, then

$(0, x, 1) - (1, 0, 0) - (0, 1, 0) - (x, 0, 1) - (y, y, 0) - (0, x, 1)$ is an induced odd cycle of length 5 in $\Gamma(R)$, where $xy = 0$ for some $x, y \in Z(A)^*$, a contradiction. □

**Theorem 2.7.** *Let $A$ be a commutative ring with $1 \neq 0$ and $R = A \times A$. If $\Gamma(A)$ contains a path on four vertices, then $ZD(R)$ is not perfect.*

*Proof.* Assume that $a - b - c - d$ is a path on four vertices. It is easy to check that $x_1 - x_2 - x_3 - x_4 - x_5 - x_1$ is an induced odd cycle of length 5 in $ZD(R)$, where $x_1 = (a, 0), x_2 = (b, -c), x_3 = (c, b), x_4 = (d, 0), x_5 = (0, 1)$.

So $ZD(R)$ is not perfect by Theorem 2.2. □

## 3. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

## References

[1] D. F. Anderson, A. Badawi, *On the total graph of a commutative ring without the zero element*, J. Algebra Appl. 11, 1250074 (2012) [18 pages].

[2] D. F. Anderson, P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434–447.

[3] D. F. Anderson, On the diameter and girth of a zero-divisor graph, II. *Houston J. Math*, (2008) 34:361-371.

[4] A. Badawi, *On the Annihilator Graph of a commutative ring*, Communications in Algebra, 42 (2014) 108-121.

[5] A. Badawi, *On the dot product graph of a commutative ring*, Communications in Algebra, 43 (2015) 43-50.

[6] R. Diestel, *Graph Theory* NY, USA: Springer-Verlag, (2000).

[7] D.B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River (2001).

**Nazi. Abachi**

Department of Mathematics, Islamic Azad University, Central Tehran

Branch, P. O. Box 14168-94351, Iran.

n_abachi@yahoo.com

**Shervin. Sahebi**

Department of Mathematics, Islamic Azad University, Central Tehran

Branch, P. O. Box 14168-94351, Iran.

sahebi@iauctb.ac.ir