Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 6 No. 1 (2019) pp 99-102.

AN ELEMENTARY PROOF OF NAGEL-SCHENZEL FORMULA

ALIREZA VAHIDI

Abstract. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finitely generated R–module, and $a_1, \ldots, a_n$ an $\mathfrak{a}$–filter regular M–sequence. The formula

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \begin{cases} \mathrm{H}^i_{(a_1,\ldots,a_n)}(M) & \text{for all i} < \text{n}, \\ \mathrm{H}^{i-n}_{\mathfrak{a}}(\mathrm{H}^n_{(a_1,\ldots,a_n)}(M)) & \text{for all i} \geq \text{n}, \end{cases}$$

is known as Nagel-Schenzel formula and is a useful result to express the local cohomology modules in terms of filter regular sequences. In this paper, we provide an elementary proof to this formula.

1. Introduction

Throughout R will denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ two ideals of R, X an arbitrary R–module which is not necessarily finitely generated, and M a finitely generated R–module. Recall that the i–th local cohomology functor $\mathrm{H}^i_{\mathfrak{a}}$ is the i–th right derived functor of the $\mathfrak{a}$–torsion functor $\Gamma_{\mathfrak{a}}$. For basic results, notations and terminology not given in this paper, the reader is referred to [1], [2], and [5].

DOI: 10.29252/as.2019.1359

MSC(2010): Primary: 13D45, Secondary: 14B15

Keywords: Filter regular sequences, local cohomology modules, Nagel-Schenzel formula.

Received: 25 Sep 2017, Accepted:20 April 2019.

This research was in part supported by a grant from Payame Noor University.

The concept of an $\mathfrak{a}$–filter regular sequence is a generalization of the concept of a filter regular sequence which has been studied in [6] and [7], and has led to some interesting results. Let $a_1, \ldots, a_n \in \mathfrak{a}$. Recall that $a_1, \ldots, a_n$ is an $\mathfrak{a}$–filter regular M–sequence if

$$\mathrm{Supp}_R\left(\frac{(a_1, \ldots, a_{i-1})M :_M a_i}{(a_1, \ldots, a_{i-1})M}\right) \subseteq \mathrm{Var}(\mathfrak{a})$$

for all $1 \leq i \leq n$, where $\mathrm{Var}(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. Let $a_1, \ldots, a_n$ be an $\mathfrak{a}$–filter regular M–sequence. Then, by [3, Proposition 1.2], we have

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \begin{cases} \mathrm{H}^i_{(a_1,\ldots,a_n)}(M) & \text{for all } i < n, \\ \mathrm{H}^{i-n}_{\mathfrak{a}}(\mathrm{H}^n_{(a_1,\ldots,a_n)}(M)) & \text{for all } i \geq n, \end{cases} \tag{1}$$

which is known as Nagel-Schenzel formula. This formula was first obtained by Nagel and Schenzel, in [4, Lemma 3.4], in the case where R is a local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{a} = \mathfrak{m}$. Both of them used the Grothendieck spectral sequence

$$\mathrm{E}^{p,q}_2 := \mathrm{H}^p_{\mathfrak{a}}(\mathrm{H}^q_{(a_1,\ldots,a_n)}(M)) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}_{\mathfrak{a}}(M)$$

to prove (1). In this paper, we provide an elementary proof to this formula.

2. **An elementary proof of** (1)

The following lemmas are needed in our proof of Nagel-Schenzel formula.

Lemma 2.1. *Let t be a non-negative integer such that $\mathrm{H}^{t-i}_{\mathfrak{a}}(\mathrm{H}^i_{\mathfrak{b}}(X)) = 0$ for all $0 \leq i \leq t$. Then $\mathrm{H}^t_{\mathfrak{a}+\mathfrak{b}}(X) = 0$.*

Proof. We prove by using induction on t. The case $t = 0$ is clear because $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(X)) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(X)$. Suppose that $t > 0$ and that $t-1$ is settled. Assume that $\overline{X} = X/\Gamma_{\mathfrak{b}}(X)$ and $Q = \mathrm{E}_R(\overline{X})/\overline{X}$ where $\mathrm{E}_R(\overline{X})$ is an injective hull of $\overline{X}$. Since $\Gamma_{\mathfrak{b}}(\overline{X}) = 0 = \Gamma_{\mathfrak{a}+\mathfrak{b}}(\overline{X})$, $\Gamma_{\mathfrak{b}}(\mathrm{E}_R(\overline{X})) = 0 = \Gamma_{\mathfrak{a}+\mathfrak{b}}(\mathrm{E}_R(\overline{X}))$. Applying the derived functors of $\Gamma_{\mathfrak{b}}(-)$ and $\Gamma_{\mathfrak{a}+\mathfrak{b}}(-)$ to the short exact sequence

$$0 \longrightarrow \overline{X} \longrightarrow \mathrm{E}_R(\overline{X}) \longrightarrow Q \longrightarrow 0,$$

we obtain the isomorphisms

$$\mathrm{H}^i_{\mathfrak{b}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{b}}(X) \tag{2}$$

and

$$\mathrm{H}^i_{\mathfrak{a}+\mathfrak{b}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \tag{3}$$

for all $i \geq 0$. From the isomorphisms (2), for all $0 \leqslant i \leqslant t-1$, we have

$$\mathrm{H}^{(t-1)-i}_{\mathfrak{a}}(\mathrm{H}^i_{\mathfrak{b}}(Q)) \cong \mathrm{H}^{t-(i+1)}_{\mathfrak{a}}(\mathrm{H}^{i+1}_{\mathfrak{b}}(X))$$

which is zero by the assumptions. Thus, from the induction hypothesis on Q, we have $\mathrm{H}^{t-1}_{\mathfrak{a}+\mathfrak{b}}(Q) = 0$. Therefore $\mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) = 0$ by the isomorphisms (3). Now, by the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X)) \longrightarrow \mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(X) \longrightarrow \mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \longrightarrow \cdots .$$

Since $\mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X)) = \mathrm{H}^{t}_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(X)) = 0$, the above long exact sequence shows that $\mathrm{H}^{t}_{\mathfrak{a}+\mathfrak{b}}(X) = 0$. □

Lemma 2.2. *Let s and t be non-negative integers such that*

(i) $\mathrm{H}^{s+t-i}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(X)) = 0$ *for all* $i \neq t$,

(ii) $\mathrm{H}^{s+t-i+1}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(X)) = 0$ *for all* $i < t$, *and*

(iii) $\mathrm{H}^{s+t-i-1}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(X)) = 0$ *for all* $i > t$.

Then we have the isomorphism $\mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t}_{\mathfrak{b}}(X)) \cong \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X)$.

Proof. Let $\overline{X} = X/\Gamma_{\mathfrak{b}}(X)$ and $Q = \mathrm{E}_R(\overline{X})/\overline{X}$ where $\mathrm{E}_R(\overline{X})$ is an injective hull of $\overline{X}$. We prove by using induction on t. In the case that $t = 0$, we have $\mathrm{H}^{s-1}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) = 0 = \mathrm{H}^{s}_{\mathfrak{a}+\mathfrak{b}}(\overline{X})$ from hypothesis (iii) and (i), and Lemma 2.1. Since $\mathrm{H}^{s}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X)) \cong \mathrm{H}^{s}_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(X))$, the assertion follows by the exact sequence

$$\mathrm{H}^{s-1}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \longrightarrow \mathrm{H}^{s}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X)) \longrightarrow \mathrm{H}^{s}_{\mathfrak{a}+\mathfrak{b}}(X) \longrightarrow \mathrm{H}^{s}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}),$$

obtained from the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0.$$

Suppose that $t > 0$ and that $t-1$ is settled. From the isomorphisms (2) and the assumptions, we have

- $\mathrm{H}^{s+(t-1)-i}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(Q)) = \mathrm{H}^{s+t-(i+1)}_{\mathfrak{a}}(\mathrm{H}^{i+1}_{\mathfrak{b}}(X)) = 0$ for all $i \neq t-1$,
- $\mathrm{H}^{s+(t-1)+1-i}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(Q)) = \mathrm{H}^{s+t+1-(i+1)}_{\mathfrak{a}}(\mathrm{H}^{i+1}_{\mathfrak{b}}(X)) = 0$ for all $i < t-1$, and
- $\mathrm{H}^{s+(t-1)-1-i}_{\mathfrak{a}}(\mathrm{H}^{i}_{\mathfrak{b}}(Q)) = \mathrm{H}^{s+t-1-(i+1)}_{\mathfrak{a}}(\mathrm{H}^{i+1}_{\mathfrak{b}}(X)) = 0$ for all $i > t-1$.

Thus we get $\mathrm{H}^{s+(t-1)}_{\mathfrak{a}+\mathfrak{b}}(Q) \cong \mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t-1}_{\mathfrak{b}}(Q))$ by the induction hypothesis on Q. Therefore $\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \cong \mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t}_{\mathfrak{b}}(X))$ from the isomorphisms (2) and (3). On the other hand, by assumptions (i) and (ii), and the exact sequence

$$\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X)) \longrightarrow \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X) \longrightarrow \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \longrightarrow \mathrm{H}^{s+t+1}_{\mathfrak{a}+\mathfrak{b}}(\Gamma_{\mathfrak{b}}(X))$$

obtained from the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0,$$

we get $\mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(\overline{X}) \cong \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X)$. Hence $\mathrm{H}^{s}_{\mathfrak{a}}(\mathrm{H}^{t}_{\mathfrak{b}}(X)) \cong \mathrm{H}^{s+t}_{\mathfrak{a}+\mathfrak{b}}(X)$ which completes the proof. □

Lemma 2.3. *Let $a_1, \ldots, a_n$ be an $\mathfrak{a}$–filter regular M–sequence. Then, for all $0 \leq i \leq n-1$, $\mathrm{Supp}_R(\mathrm{H}^i_{(a_1,\ldots,a_n)}(M)) \subseteq \mathrm{Var}(\mathfrak{a})$. In particular,*

$$\mathrm{H}^j_{\mathfrak{a}}(\mathrm{H}^i_{(a_1,\ldots,a_n)}(M)) \cong \begin{cases} \mathrm{H}^i_{(a_1,\ldots,a_n)}(M) & \text{if } j=0, \\ 0 & \text{if } j>0, \end{cases}$$

for all $0 \leq i \leq n-1$.

Proof. Let $0 \leq i \leq n-1$ and $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{H}^i_{(a_1,\ldots,a_n)}(M))$. Assume contrarily that $\mathfrak{p} \notin \mathrm{Var}(\mathfrak{a})$. Thus $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$ and so $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ is a weak $M_{\mathfrak{p}}$–sequence. Hence $\mathrm{H}^i_{(\frac{a_1}{1},\ldots,\frac{a_n}{1})}(M_{\mathfrak{p}}) = 0$. Therefore we get $(\mathrm{H}^i_{(a_1,\ldots,a_n)}(M))_{\mathfrak{p}} = 0$. This contradiction shows that $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$. □

Now we are ready to give an elementary and simple proof for (1).

Proof of Nagel-Schenzel formula. Let $i < n$ (resp. $i \geq n$). Consider Lemma 2.3 and apply Lemma 2.2 with $s = 0$, $t = i$, and $\mathfrak{b} = (a_1, \ldots, a_n)$ (resp. $s = i-n$, $t = n$, and $\mathfrak{b} = (a_1, \ldots, a_n)$).

References

[1] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge, 1998.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1998.

[3] K. Khashyarmanesh and Sh. Salarian, *Filter regular sequences and the finiteness of local cohomology modules*, Comm. Algebra **26** (1998) 2483–2490.

[4] U. Nagel and P. Schenzel, *Cohomological annihilators and Castelnuovo-Mumford regularity, Commutative algebra: Syzygies, multiplicities, and birational algebra*, Contemp. Math. **159** (1994) 307–328.

[5] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, San Diego, 1979.

[6] P. Schenzel, N. V. Trung, and N. T. Cuong, *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr. **85** (1978) 57–73.

[7] J. Stuckrad and W. Vogel, *Buchsbaum Rings and Applications*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.

Alireza Vahidi

Department of Mathematics,

Payame Noor University (PNU),

P.O.BOX, 19395-4697,

Tehran, Iran.

vahidi.ar@pnu.ac.ir