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SPECTRA OF SOME NEW EXTENDED CORONA

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Abstract. For two graphs G and H with n and m vertices, the corona $G \circ H$ of G and H is the graph obtained by taking one copy of G and n copies of H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The neighborhood corona $G \star H$ of G and H is the graph obtained by taking one copy of G and n copies of H and joining every neighbor of the i^{th} vertex of G to every vertex in the i^{th} copy of H. In this paper, we define four new extensions of corona and neighborhood corona of two graphs G and H; named the identity-extended corona, identity-extended neighborhood corona, neighborhood extended corona and neighborhood extended neighborhood corona and then determine the spectrum of their adjacency matrix, where H is a regular graph. As an application, we exhibit infinite families of integral graphs.

1. Introduction

All graphs considered in this paper are undirected and simple. Let G be a graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and edge set E(G). Let $M = M(G)$ be a $n \times n$ matrix associated to G, the set of all the eigenvalues of M is called the M-spectrum of graph G which are the roots

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of the characteristic polynomial of M. In this paper, the characteristic polynomial of M is denoted by $\varphi(\mathrm{M}; x) = det(x\mathrm{I}_n - \mathrm{M})$, where I_n is the identity matrix of size n. The adjacency matrix of G, denoted by $\mathrm{A(G)} = (a_{ij})_{n\times n}$, is a $n \times n$ symmetric matrix in which $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. The graph G is integral if A-spectrum of G consists of only integers. K_n, $\mathrm{K}_{n,n}$ and C_n (just for $n = 3, 4, 6$) are examples of integral graphs; see [3] for more details. Let $d_i = d_{\mathrm{G}}(v_i)$ be the degree of vertex v_i in G and $\mathrm{D(G)} = (d_{ij})_{n\times n}$ be the diagonal matrix such that $d_{ii} = d_i$ for each $1 \leq i \leq n$. The Laplacian matrix and the signless Laplacian matrix of G are defined as $\mathrm{L(G)} = \mathrm{D(G)} - \mathrm{A(G)}$ and $\mathrm{Q(G)} = \mathrm{D(G)} + \mathrm{A(G)}$, respectively.

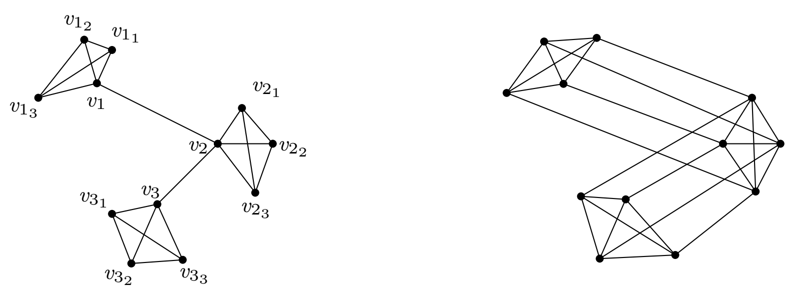
So far, the various spectra of many graph operations such as the disjoint union, Cartesian product, Kronecker product, hierarchical product and subdivision-vertex and subdivision-edge join are computed in [6, 7, 9, 10, 17, 18, 20]. In [13], Harary introduced the concept of corona of a graph. Followed by, the A-spectrum, L-spectrum and Q-spectrum of the corona, edge corona, neighborhood corona, subdivision-vertex and subdivision-edge corona and subdivision-vertex and subdivision-edge neighborhood corona of two graphs G and H were computed in [4, 8, 12, 15, 19, 21, 22, 24]. Barik and Sahoo [5] have defined some more variations of double corona graphs and described their Laplacian spectrum. Recently, Adiga et al. [1, 2] have introduced some new extension of corona, neighborhood corona and edge corona of two graphs and computed their A-spectrum, L-spectrum and Q-spectrum. In [23], an extended corona product for weighted graphs is defined and then applying this generalized corona product and the reinforcement mechanism of edge weight in realistic networks, the authors introduced a simple generative model for heterogeneous weighted networks, which leads to rich topological and weighted properties.

In this paper, motivated by [1], we define four new extended corona of two graphs G and H and determine their A-spectrum where H is a regular graph. As an application, we exhibit infinite families of integral graphs.

2. Preliminaries

In this section, we need to state some results which will be used frequently later. Let G and H be two graphs on n and m vertices such that $\mathrm{V(G)} = \{v_1, v_2, ..., v_n\}$ and $\mathrm{V(H)} = \{u_1, u_2, ..., u_m\}$. The corona $\mathrm{G} \circ \mathrm{H}$ of G and H is the graph obtained by taking one copy of G and n copies of H and if $\mathrm{V(G \circ H)} = \{v_{1_1}, ..., v_{1_m}, v_{2_1}, ..., v_{2_m}, ..., v_{n_1}, ..., v_{n_m}, v_1, ..., v_n\}$, then for each $i \in \{1, 2, ..., n\}$, v_i is adjacent to v_{i_k} for each $k \in \{1, 2, ..., m\}$ where v_{i_k} is considered the same as u_k in the i^{th} copy of H. The neighborhood corona $\mathrm{G} \star \mathrm{H}$ of two graphs G and H is the graph obtained by taking one copy of G and n copies of H and if $\mathrm{V(G \star H)} = \{v_{1_1}, ..., v_{1_m}, v_{2_1}, ..., v_{2_m}, ..., v_{n_1}, ..., v_{n_m}, v_1, ..., v_n\}$, then for each $i \in \{1, 2, ..., n\}$, every neighbor of v_i is adjacent to v_{i_k} for each $k \in \{1, 2, ..., m\}$, again v_{i_k} is considered the same as u_k in the i^{th} copy of H. Let σ be a permutation of n items $\{1, 2, ..., n\}$ which is written in the one-line notation as $\sigma(1), \sigma(2), ..., \sigma(n)$, that is, an ordered arrangement of $\{1, 2, ..., n\}$. If for each $i \in \{1, 2, ..., n\}$, $\sigma(i) = i$, we call σ the identity permutation. Now, motivated by [1] and the concept of permutation, we define the identity-extended corona and identity-extended neighborhood corona as follows:

Definition 2.1. The identity-extended corona $I_{ex}(\mathrm{G} \circ \mathrm{H})$ of two graphs G and H is the graph obtained by taking the corona $\mathrm{G} \circ \mathrm{H}$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertex v_{j_k} of j^{th} copy of H, provided the vertices v_i and v_j are adjacent in G; see Figure 1.



$\mathrm{P}_3 \circ \mathrm{C}_3$ $\quad$ $I_{ex}(\mathrm{P}_3 \circ \mathrm{C}_3)$

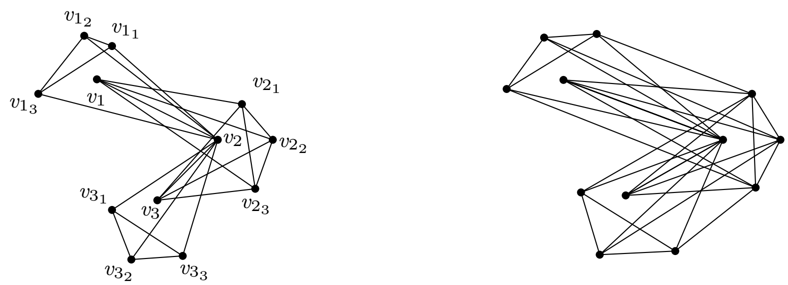
FIGURE 1. the corona and the identity-extended corona of two graphs P_3 and C_3

Definition 2.2. The identity-extended neighborhood corona $I_{ex}(\mathrm{G} \star \mathrm{H})$ of two graphs G and H is the graph obtained by taking the neighborhood corona $\mathrm{G} \star \mathrm{H}$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertex v_{j_k} of j^{th} copy of H, provided the vertices v_i and v_j are adjacent in G; see Figure 2.

The following definitions are two other extensions of corona and neighborhood corona of graphs.

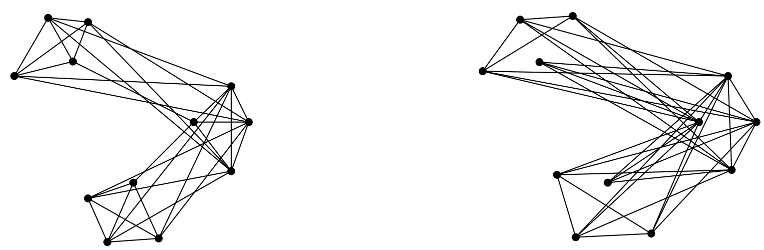
Definition 2.3. The neighborhood extended corona $N_{ex}(\mathrm{G} \circ \mathrm{H})$ of two graphs G and H is the graph obtained by taking the corona $\mathrm{G} \circ \mathrm{H}$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertices v_{j_l} of j^{th} copy of H, provided the vertices v_i and v_j are adjacent in G and u_k and u_l are adjacent in H; see Figure 3.

Definition 2.4. The neighborhood extended neighborhood corona $N_{ex}(\mathrm{G} \star \mathrm{H})$ of two graphs G and H is the graph obtained by taking the neighborhood corona $\mathrm{G} \star \mathrm{H}$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertices v_{j_l} of j^{th} copy of H, provided the vertices v_i and v_j are adjacent in G and u_k and u_l are adjacent in H; see Figure 3.



$P_3 \star C_3$ $\qquad$ $I_{ex}(P_3 \star C_3)$

FIGURE 2. the neighborhood corona and the identity-extended neighborhood corona of two graphs P_3 and C_3



$N_{ex}(P_3 \circ C_3)$ $\qquad$ $N_{ex}(P_3 \star C_3)$

FIGURE 3. Two neighborhood extended corona of two graphs P_3 and C_3

In this paper, suppose that $J = [e_{ij}]_{s\times t}$ denotes the $s \times t$ matrix with all entries equal to one and $J'_{s\times t}$ obtained by replacing every entry of $J_{s\times t}$ by 0 except e_{11}. The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{p\times q}$ is the $np \times mq$ matrix obtained from A by replacing each entry a_{ij} by $a_{ij}B$, see [14]. For matrices A, B, C and D, $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist.

The M-coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1}\mathbf{1}_n,$$

where $\mathbf{1}_n$ denotes the column vector of dimension n with all the components equal to one; see [22].

Proposition 2.5 ([8]). *Let* M *be an* $n \times n$ *matrix with each row sum equal to constant* c, *then*

$$\Gamma_M(x) = \frac{n}{x - c}.$$

Proposition 2.6 ([20]). *Let* A *be an* $n \times n$ *real matrix, then*

$$det(x\mathrm{I}_n - \mathrm{A} - \alpha \mathrm{J}_{n\times n}) = (1 - \alpha\Gamma_{\mathrm{A}}(x)) \cdot det(x\mathrm{I}_n - \mathrm{A}).$$

Proposition 2.7 ([25]). *Let* M_1, M_2, M_3 *and* M_4 *be respectively* $p \times p$, $p \times q$, $q \times p$ *and* $q \times q$ *matrices with* M_1 *and* M_4 *invertible. Then*

$$\begin{aligned} det(\begin{bmatrix} \mathrm{M}_1 & \mathrm{M}_2 \\ \mathrm{M}_3 & \mathrm{M}_4 \end{bmatrix}) &= det(\mathrm{M}_4) \cdot det(\mathrm{M}_1 - \mathrm{M}_2\mathrm{M}_4^{-1}\mathrm{M}_3) \\ &= det(\mathrm{M}_1) \cdot det(\mathrm{M}_4 - \mathrm{M}_3\mathrm{M}_1^{-1}\mathrm{M}_2), \end{aligned}$$

where $\mathrm{M}_1 - \mathrm{M}_2\mathrm{M}_4^{-1}\mathrm{M}_3$ *and* $\mathrm{M}_4 - \mathrm{M}_3\mathrm{M}_1^{-1}\mathrm{M}_2$ *are called the Schur complements of* M_4 *and* M_1*, respectively.*

In this paper, we apply the Laplace's expansion of a determinants by complementry minors several times; For more details, one can see page 36 of [11].

3. Spectra of the identity-extended corona and identity-extended neighborhood corona

In this section, G is an arbitrary graph with n vertices and H is a r-regular graph with m vertices. To write the adjacency matrix of the identity-extended corona and identity-extended neighborhood corona, we maintain the arrangement of vertices in $\mathrm{V}(\mathrm{G} \circ \mathrm{H})$. First, we present the spectra of $I_{ex}(\mathrm{G} \circ \mathrm{H})$.

Theorem 3.1. *The adjacency spectrum of* $I_{ex}(\mathrm{G} \circ \mathrm{H})$ *is:*
i) $\lambda_i(\mathrm{H}) + \lambda_j(\mathrm{G})$ *for* $i = 2, ..., m$ *and* $j = 1, ..., n$*;*
ii) $(2\lambda_j(\mathrm{G}) + r \pm \sqrt{r^2 + 4m})/2$ *for* $j = 1, ..., n$.

Proof. The adjacency matrix of $I_{ex}(\mathrm{G} \circ \mathrm{H})$ can be expressed in the following form:

$$\mathrm{A}(I_{ex}(\mathrm{G} \circ \mathrm{H})) = \begin{bmatrix} \mathrm{I}_n \otimes \mathrm{A}(\mathrm{H}) + \mathrm{A}(\mathrm{G}) \otimes \mathrm{I}_m & \mathrm{I}_n \otimes \mathrm{J}_{m\times 1} \\ \mathrm{I}_n \otimes \mathrm{J}_{1\times m} & \mathrm{A}(\mathrm{G}) \end{bmatrix}.$$

A(H) is a real Hermitian matrix, so A(H) is orthogonally diagonalizable and Since H is r-regular, $\mathrm{A}(\mathrm{H}) = \mathrm{P}\mathrm{D}(\mathrm{H})\mathrm{P}^{\mathrm{T}}$ where $\mathrm{D}(\mathrm{H}) = diag(r, \lambda_2(\mathrm{H}), ..., \lambda_m(\mathrm{H}))$ and P is a square matrix of order m such that its first column vector is $\frac{1}{\sqrt{m}}(1,1,...,1)$ and $\mathrm{P^TP} = \mathrm{I_m}$. Therefore,

$$\begin{aligned}
\mathrm{A}(I_{ex}(\mathrm{G}\circ\mathrm{H})) &= \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{PD(H)P^T} + \mathrm{A(G)}\otimes \mathrm{I}_m & \mathrm{I}_n\otimes \mathrm{J}_{m\times 1} \\ \mathrm{I}_n\otimes \mathrm{J}_{1\times m} & \mathrm{A(G)} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{P} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{D(H)} + \mathrm{A(G)}\otimes \mathrm{I}_m & \mathrm{I}_n\otimes \mathrm{P^T J}_{m\times 1} \\ \mathrm{I}_n\otimes \mathrm{J}_{1\times m}\mathrm{P} & \mathrm{A(G)} \end{bmatrix} \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{P^T} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{P} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{D(H)} + \mathrm{A(G)}\otimes \mathrm{I}_m & \mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{m\times 1} \\ \mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{1\times m} & \mathrm{A(G)} \end{bmatrix} \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{P^T} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}.
\end{aligned}$$

So $\mathrm{A}(I_{ex}(\mathrm{G}\circ\mathrm{H}))$ is similar to

$$\mathrm{B}_0 = \begin{bmatrix} \mathrm{I}_n\otimes \mathrm{D(H)} + \mathrm{A(G)}\otimes \mathrm{I}_m & \mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{m\times 1} \\ \mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{1\times m} & \mathrm{A(G)} \end{bmatrix},$$

and

$$\begin{aligned}
det(x\mathrm{I} - \mathrm{A}(I_{ex}(\mathrm{G}\circ\mathrm{H}))) &= det(x\mathrm{I} - \mathrm{B}_0) \\
&= det \begin{bmatrix} \mathrm{I}_n\otimes (x\mathrm{I}_m - \mathrm{D(H)}) - \mathrm{A(G)}\otimes \mathrm{I}_m & -\mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{m\times 1} \\ -\mathrm{I}_n\otimes \sqrt{m}\mathrm{J'}_{1\times m} & x\mathrm{I}_n - \mathrm{A(G)} \end{bmatrix}.
\end{aligned}$$

Using the Laplace's expansion of $det(x\mathrm{I} - \mathrm{B}_0)$ by $(mi+2), (mi+3), ..., (mi+m)^{th}$ columns for $i = 0, 1, ..., n-1$, we see that $det(x\mathrm{I} - \mathrm{B}_0) = \mathrm{M.M}_1$ where $\mathrm{M} = det(\mathrm{B}_1)$ and $\mathrm{B}_1 = \mathrm{I}_n \otimes diag(x - \lambda_2(\mathrm{H}), ..., x - \lambda_m(\mathrm{H})) - (\mathrm{A(G)} \otimes \mathrm{I}_{m-1})$, and also

$$\mathrm{M}_1 = det \begin{bmatrix} (x-r)\mathrm{I}_n - \mathrm{A(G)} & -\sqrt{m}\mathrm{I}_n \\ -\sqrt{m}\mathrm{I}_n & x\mathrm{I}_n - \mathrm{A(G)} \end{bmatrix}$$

is the complementary minor of M. Again expanding $det(\mathrm{B}_1)$ by Laplace 's method in the similar way and continuing this method $(m-2)$ times, we have

$$\begin{aligned}
det(\mathrm{B}_1) &= \prod_{i=2}^{m} det((x - \lambda_i(\mathrm{H}))\mathrm{I}_n - \mathrm{A(G)}) \\
&= \prod_{i=2}^{m}\prod_{j=1}^{n} (x - \lambda_i(\mathrm{H}) - \lambda_j(\mathrm{G})) \\
&= \prod_{\substack{2\leqslant i\leqslant m \\ 1\leqslant j\leqslant n}} (x - (\lambda_i(\mathrm{H}) + \lambda_j(\mathrm{G}))).
\end{aligned}$$

Again A(G) is orthogonally diagonalizable, so we can easily see that

$$\mathrm{M}_1 = det \begin{bmatrix} (x-r)\mathrm{I}_n - \mathrm{D(G)} & -\sqrt{m}\mathrm{I}_n \\ -\sqrt{m}\mathrm{I}_n & x\mathrm{I}_n - \mathrm{D(G)} \end{bmatrix}.$$

By Proposition 2.7,

$$\begin{aligned}
\mathrm{M}_1 &= det(x\mathrm{I}_n - \mathrm{D(G)}) \cdot det((x-r)\mathrm{I}_n - \mathrm{D(G)} - m(x\mathrm{I}_n - \mathrm{D(G)})^{-1}) \\
&= det(x\mathrm{I}_n - \mathrm{D(G)}) \cdot \prod_{i=1}^{n} (x - r - \lambda_i(\mathrm{G}) - \frac{m}{x-\lambda_i(\mathrm{G})}) \\
&= \prod_{i=1}^{n} (x - \lambda_i(\mathrm{G})) \cdot \prod_{i=1}^{n} \frac{x^2 - (2\lambda_i(\mathrm{G}) + r)x + \lambda_i^2(\mathrm{G}) + r\lambda_i(\mathrm{G}) - m}{x - \lambda_i(\mathrm{G})} \\
&= \prod_{i=1}^{n} (x^2 - (2\lambda_i(\mathrm{G}) + r)x + (\lambda_i^2(\mathrm{G}) + r\lambda_i(\mathrm{G}) - m)).
\end{aligned}$$

Finally,

$$\begin{aligned}
det(x\mathrm{I} - \mathrm{A}(I_{ex}(\mathrm{G} \circ \mathrm{H}))) = &\prod_{\substack{2 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} (x - (\lambda_i(\mathrm{H}) + \lambda_j(\mathrm{G}))) \\
&\cdot \prod_{i=1}^{n} (x^2 - (2\lambda_i(\mathrm{G}) + r)x + (\lambda_i^2(\mathrm{G}) + r\lambda_i(\mathrm{G}) - m)).
\end{aligned}$$

□

Now, we present the spectra of the identity-extended neighborhood corona $I_{ex}(\mathrm{G} \star \mathrm{H})$ of two graphs G and H, where H is regular.

Theorem 3.2. *The adjacency spectrum of $I_{ex}(\mathrm{G} \star \mathrm{H})$ is as follows:*
i) $\lambda_i(\mathrm{H}) + \lambda_j(\mathrm{G})$ *for* $i = 2, ..., m$ *and* $j = 1, ..., n$;
ii) $(2\lambda_j(\mathrm{G}) + r \pm \sqrt{r^2 + 4m\lambda_j^2(\mathrm{G})})/2$ *for* $j = 1, ..., n$.

Proof. The adjacency matrix of $I_{ex}(\mathrm{G} \star \mathrm{H})$ can be written as:

$$\mathrm{A}(I_{ex}(\mathrm{G} \star \mathrm{H})) = \begin{bmatrix} \mathrm{I}_n \otimes \mathrm{A(H)} + \mathrm{A(G)} \otimes \mathrm{I}_m & \mathrm{A(G)} \otimes \mathrm{J}_{m\times 1} \\ \mathrm{A(G)} \otimes \mathrm{J}_{1\times m} & \mathrm{A(G)} \end{bmatrix}.$$

Similar to the proof of Theorem 3.1, $\mathrm{A}(I_{ex}(\mathrm{G} \star \mathrm{H}))$ is similar to

$$\mathrm{B}_0 = \begin{bmatrix} \mathrm{I}_n \otimes \mathrm{D(H)} + \mathrm{A(G)} \otimes \mathrm{I}_m & \mathrm{A(G)} \otimes \sqrt{m}\mathrm{J}'_{m\times 1} \\ \mathrm{A(G)} \otimes \sqrt{m}\mathrm{J}'_{1\times m} & \mathrm{A(G)} \end{bmatrix},$$

and

$$\begin{aligned}
det(x\mathrm{I} - \mathrm{A}(I_{ex}(\mathrm{G} \star \mathrm{H}))) &= det(x\mathrm{I} - \mathrm{B}_0) \\
&= det \begin{bmatrix} \mathrm{I}_n \otimes (x\mathrm{I}_m - \mathrm{D(H)}) - \mathrm{A(G)} \otimes \mathrm{I}_m & -\mathrm{A(G)} \otimes \sqrt{m}\mathrm{J}'_{m\times 1} \\ -\mathrm{A(G)} \otimes \sqrt{m}\mathrm{J}'_{1\times m} & x\mathrm{I}_n - \mathrm{A(G)} \end{bmatrix}.
\end{aligned}$$

Again similar to the proof of Theorem 3.1; using the Laplace's expansion,

$$det(x\mathrm{I}-\mathrm{A}(I_{ex}(\mathrm{G}\star\mathrm{H})))=\prod_{\substack{2\leqslant i\leqslant m\\1\leqslant j\leqslant n}}(x-(\lambda_i(\mathrm{H})+\lambda_j(\mathrm{G})))\cdot det\begin{bmatrix}(x-r)\mathrm{I}_n-\mathrm{A}(\mathrm{G}) & -\sqrt{m}\mathrm{A}(\mathrm{G})\\ -\sqrt{m}\mathrm{A}(\mathrm{G}) & x\mathrm{I}_n-\mathrm{A}(\mathrm{G})\end{bmatrix}.$$

A(G) is orthogonally diagonalizable, so we can easily see that

$$\begin{aligned}det(x\mathrm{I}-\mathrm{A}(I_{ex}(\mathrm{G}\star\mathrm{H})))&=\prod_{\substack{2\leqslant i\leqslant m\\1\leqslant j\leqslant n}}(x-(\lambda_i(\mathrm{H})+\lambda_j(\mathrm{G})))\cdot det\begin{bmatrix}(x-r)\mathrm{I}_n-\mathrm{D}(\mathrm{G}) & -\sqrt{m}\mathrm{D}(\mathrm{G})\\ -\sqrt{m}\mathrm{D}(\mathrm{G}) & x\mathrm{I}_n-\mathrm{D}(\mathrm{G})\end{bmatrix}\\&=\prod_{\substack{2\leqslant i\leqslant m\\1\leqslant j\leqslant n}}(x-(\lambda_i(\mathrm{H})+\lambda_j(\mathrm{G})))\cdot\prod_{i=1}^{n}(x^2-(2\lambda_i(\mathrm{G})+r)x+r\lambda_i(\mathrm{G})+(1-m)\lambda_i^2(\mathrm{G})).\end{aligned}$$

□

4. Spectra of the neighborhood extended corona and neighborhood extended neighborhood corona

In this section, we present the spectra of the neighborhood extended corona $N_{ex}(\mathrm{G}\circ\mathrm{H})$ and the neighborhood extended neighborhood corona $N_{ex}(\mathrm{G}\star\mathrm{H})$ of two graphs G and H where G is an arbitrary n-vertices and H is a r-regular m-vertices graph. To write the adjacency matrix of these two operations, we maintain the arrangement of vertices in $\mathrm{V}(\mathrm{G}\star\mathrm{H})$.

Theorem 4.1. *The adjacency spectrum of* $N_{ex}(\mathrm{G}\circ\mathrm{H})$ *is:*
i) $\lambda_i(\mathrm{H})(1+\lambda_j(\mathrm{G}))$ *for* $i=2,...,m$ *and* $j=1,...,n$*;*
ii) $((r+1)\lambda_j(\mathrm{G})+r\pm\sqrt{((r-1)\lambda_j(\mathrm{G})+r)^2+4m})/2$ *for* $j=1,...,n$.

Proof. The adjacency matrix of $N_{ex}(\mathrm{G}\circ\mathrm{H})$ can be expressed in the following form:

$$\mathrm{A}(N_{ex}(\mathrm{G}\circ\mathrm{H}))=\begin{bmatrix}(\mathrm{I}_n+\mathrm{A}(\mathrm{G}))\otimes\mathrm{A}(\mathrm{H}) & \mathrm{I}_n\otimes\mathrm{J}_{m\times1}\\ \mathrm{I}_n\otimes\mathrm{J}_{1\times m} & \mathrm{A}(\mathrm{G})\end{bmatrix}.$$

A(H) is a real Hermitian matrix and H is r-regular, so we have $\mathrm{A}(\mathrm{H})=\mathrm{P}\mathrm{D}(\mathrm{H})\mathrm{P}^{\mathrm{T}}$ where $\mathrm{A}(\mathrm{H})=diag(r,\lambda_2(\mathrm{H}),...,\lambda_m(\mathrm{H}))$ and P is a square matrix of order m such that its first column vector is $\frac{1}{\sqrt{m}}(1,1,...,1)$ and $\mathrm{P}^{\mathrm{T}}\mathrm{P} = \mathrm{I_m}$. Therefore,

$$\mathrm{A}(N_{ex}(\mathrm{G}\circ\mathrm{H})) = \begin{bmatrix} \mathrm{I}_n \otimes \mathrm{P} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix} \begin{bmatrix} (\mathrm{I}_n + \mathrm{A(G)}) \otimes \mathrm{D(H)} & \mathrm{I}_n \otimes \sqrt{m}\mathrm{J}'_{m\times 1} \\ \mathrm{I}_n \otimes \sqrt{m}\mathrm{J}'_{1\times m} & \mathrm{A(G)} \end{bmatrix} \begin{bmatrix} \mathrm{I}_n \otimes \mathrm{P}^{\mathrm{T}} & 0 \\ 0 & \mathrm{I}_n \end{bmatrix}.$$

So $\mathrm{A}(N_{ex}(\mathrm{G}\circ\mathrm{H}))$ is similar to

$$\mathrm{B}_0 = \begin{bmatrix} (\mathrm{I}_n + \mathrm{A(G)}) \otimes \mathrm{D(H)} & \mathrm{I}_n \otimes \sqrt{m}\mathrm{J}'_{m\times 1} \\ \mathrm{I}_n \otimes \sqrt{m}\mathrm{J}'_{1\times m} & \mathrm{A(G)} \end{bmatrix},$$

and

$$det(x\mathrm{I} - \mathrm{A}(N_{ex}(\mathrm{G}\circ\mathrm{H}))) = det(x\mathrm{I} - \mathrm{B}_0).$$

Now, use the Laplace's expansion of $det(x\mathrm{I}-\mathrm{B}_0)$ by $(mi+2),(mi+3),...,(mi+m)^{th}$ columns for $i = 0,1,...,n-1$. We have

$$det(x\mathrm{I} - \mathrm{B}_0) = det(\mathrm{B}_1)\cdot \mathrm{M}_1,$$

where $\mathrm{B}_1 = \mathrm{I}_n \otimes diag(x-\lambda_2(\mathrm{H}),...,x-\lambda_m(\mathrm{H})) - (\mathrm{A(G)} \otimes diag(\lambda_2(\mathrm{H}),...,\lambda_m(\mathrm{H})))$ and

$$\mathrm{M}_1 = det\begin{bmatrix} (x-r)\mathrm{I}_n - r\mathrm{A(G)} & -\sqrt{m}\mathrm{I}_n \\ -\sqrt{m}\mathrm{I}_n & x\mathrm{I}_n - \mathrm{A(G)} \end{bmatrix}$$

is the complementary minor of $det(\mathrm{B}_1)$. Again expanding $det(\mathrm{B}_1)$ by Laplace 's method in the similar way and continuing this method $(m-2)$ times, we have

$$\begin{aligned} det(\mathrm{B}_1) &= \prod_{i=2}^{m} det((x-\lambda_i(\mathrm{H}))\mathrm{I}_n - \lambda_i(\mathrm{H})\mathrm{A(G)}) \\ &= \prod_{\substack{2\leqslant i\leqslant m \\ 1\leqslant j\leqslant n}} (x - \lambda_i(\mathrm{H})(1+\lambda_j(\mathrm{G}))). \end{aligned}$$

A(G) is orthogonally diagonalizable, so we can easily see that

$$\mathrm{M}_1 = det\begin{bmatrix} (x-r)\mathrm{I}_n - r\mathrm{D(G)} & -\sqrt{m}\mathrm{I}_n \\ -\sqrt{m}\mathrm{I}_n & x\mathrm{I}_n - \mathrm{D(G)} \end{bmatrix}.$$

By Proposition 2.7,

$$\begin{aligned}
\mathrm{M}_1 &= det(x\mathrm{I}_n - \mathrm{D(G)}).det((x-r)\mathrm{I}_n - r\mathrm{D(G)} - m(x\mathrm{I}_n - \mathrm{D(G)})^{-1}) \\
&= det(x\mathrm{I}_n - \mathrm{D(G)}) \cdot \prod_{i=1}^{n}(x - r - r\lambda_i(\mathrm{G}) - \frac{m}{x-\lambda_i(\mathrm{G})}) \\
&= \prod_{i=1}^{n}(x-\lambda_i(\mathrm{G})). \prod_{i=1}^{n} \frac{x^2 - ((r+1)\lambda_i(\mathrm{G}) + r)x + r\lambda_i^2(\mathrm{G}) + r\lambda_i(\mathrm{G}) - m}{x - \lambda_i(\mathrm{G})} \\
&= \prod_{i=1}^{n}(x^2 - ((r+1)\lambda_i(\mathrm{G}) + r)x + (r\lambda_i^2(\mathrm{G}) + r\lambda_i(\mathrm{G}) - m)).
\end{aligned}$$

□

The proof of the next theorem is omitted because of similarity.

Theorem 4.2. *The adjacency spectrum of* $N_{ex}(\mathrm{G} \star \mathrm{H})$ *is:*
i) $\lambda_i(\mathrm{H})(1 + \lambda_j(\mathrm{G}))$ *for* $i = 2, ..., m$ *and* $j = 1, ..., n$;
ii) $((r+1)\lambda_j(\mathrm{G}) + r \pm \sqrt{((r-1)\lambda_j(\mathrm{G}) + r)^2 + 4m\lambda_j^2(\mathrm{G})})/2$ *for* $j = 1, ..., n$.

5. New classes of integral graphs

In this section, we construct infinitely many families of integral graphs. The next theorem follows from Theorem 3.1, 3.2, 4.1 and 4.2.

Theorem 5.1. *suppose that* G *is an integral graph with* n *vertices and* H *is a* r*-regular integral graph with* m *vertices, then*

(1) *The graph* $I_{ex}(\mathrm{G} \circ \mathrm{H})$ *is integral if* $m = h(h \pm r)$ *for some positive integer* h.

(2) *The graph* $I_{ex}(\mathrm{G} \star \mathrm{H})$ *is integral if for every* $j = 1, 2, \cdots, n$, $m\lambda_j^2(\mathrm{G}) = h(h \pm r)$ *for some positive integer* h.

(3) $N_{ex}(\mathrm{G} \circ p\mathrm{K}_2)$ *is integral graph if* $8p + 1$ *is perfect square.*

(4) $N_{ex}(\mathrm{G} \star p\mathrm{K}_1)$ *is integral graph if* $4p + 1$ *is perfect square.*

As an special case of previous theorem, we have the following corollary.

Corollary 5.2. *suppose that* G *is an integral graph, then for each positive integer* p *and* q,

(1) $I_{ex}(\mathrm{G} \circ \mathrm{K}_p)$, $I_{ex}(\mathrm{G} \circ p\mathrm{K}_{p,p})$ *and* $I_{ex}(\mathrm{G} \circ \mathrm{C}_3)$ *are integral graphs.*

(2) $I_{ex}(q\mathrm{K}_2 \star \mathrm{K}_p)$ *and* $I_{ex}(q\mathrm{K}_2 \star p\mathrm{K}_{p,p})$ *are integral graphs.*

Remark 5.3. In Theorem 5.1 and Corollary 5.2, one can replace G with any graph of the classes of integral graphs has obtained so far. in fact, all the integral graphs obtained from new operations on the graphs can replace by G; for example, one can see the class of integral graphs obtained in [1, 2, 16, 17].

6. Question

Regarding the concept of permutation and definition of identity-extended corona and identity-extended neighborhood corona, it seems that this concept can be generalized to the σ-extended corona and σ-extended neighborhood corona. It seems that such an extension of the corona corresponds to many molecular bonds, and thus by obtaining a variety of spectra of such expansion, some of the indices associated with those bonds would be calculated. But this question still remains open, how can obtain these spectrums?

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