SPECTRA OF SOME NEW EXTENDED CORONA

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ABSTRACT. For two graphs G and H with n and m vertices, the corona $G \circ H$ of G and H is the graph obtained by taking one copy of G and n copies of H and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H . The neighborhood corona $G \star H$ of G and H is the graph obtained by taking one copy of G and n copies of H and joining every neighbor of the i^{th} vertex of G to every vertex in the i^{th} copy of H . In this paper, we define four new extensions of corona and neighborhood corona of two graphs G and H ; named the identity-extended corona, identity-extended neighborhood corona, neighborhood extended corona and neighborhood extended neighborhood corona and then determine the spectrum of their adjacency matrix, where H is a regular graph. As an application, we exhibit infinite families of integral graphs.

1. INTRODUCTION

All graphs considered in this paper are undirected and simple. Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. Let $M = M(G)$ be a $n \times n$ matrix associated to G ,

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the set of all the eigenvalues of M is called the M -spectrum of graph G which are the roots of the characteristic polynomial of M . In this paper, the characteristic polynomial of M is denoted by $\varphi(M; x) = \det(xI_n - M)$, where I_n is the identity matrix of size n . The adjacency matrix of G , denoted by $A(G) = (a_{ij})_{n \times n}$, is a $n \times n$ symmetric matrix in which $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. The graph G is integral if A -spectrum of G consists of only integers. K_n , $K_{n,n}$ and C_n (just for $n = 3, 4, 6$) are examples of integral graphs; see [3] for more details. Let $d_i = d_G(v_i)$ be the degree of vertex v_i in G and $D(G) = (d_{ij})_{n \times n}$ be the diagonal matrix such that $d_{ii} = d_i$ for each $1 \leq i \leq n$. The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively.

So far, the various spectra of many graph operations such as the disjoint union, Cartesian product, Kronecker product, hierarchical product and subdivision-vertex and subdivision-edge join are computed in [6, 7, 9, 10, 17, 18, 20]. In [13], Harary introduced the concept of corona of a graph. Followed by, the A -spectrum, L -spectrum and Q -spectrum of the corona, edge corona, neighborhood corona, subdivision-vertex and subdivision-edge corona and subdivision-vertex and subdivision-edge neighborhood corona of two graphs G and H were computed in [4, 8, 12, 15, 19, 21, 22, 24]. Barik and Sahoo [5] have defined some more variations of double corona graphs and described their Laplacian spectrum. Recently, Adiga et al. [1, 2] have introduced some new extension of corona, neighborhood corona and edge corona of two graphs and computed their A -spectrum, L -spectrum and Q -spectrum. In [23], an extended corona product for weighted graphs is defined and then applying this generalized corona product and the reinforcement mechanism of edge weight in realistic networks, the authors introduced a simple generative model for heterogeneous weighted networks, which leads to rich topological and weighted properties.

In this paper, motivated by [1], we define four new extended corona of two graphs G and H and determine their A -spectrum where H is a regular graph. As an application, we exhibit infinite families of integral graphs.

2. PRELIMINARIES

In this section, we need to state some results which will be used frequently later. Let G and H be two graphs on n and m vertices such that $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_m\}$. The corona $G \circ H$ of G and H is the graph obtained by taking one copy of G and n copies of H and if $V(G \circ H) = \{v_{1_1}, \dots, v_{1_m}, v_{2_1}, \dots, v_{2_m}, \dots, v_{n_1}, \dots, v_{n_m}, v_1, \dots, v_n\}$, then for each $i \in \{1, 2, \dots, n\}$, v_i is adjacent to v_{i_k} for each $k \in \{1, 2, \dots, m\}$ where v_{i_k} is considered the same as u_k in the i^{th} copy of H . The neighborhood corona $G \star H$ of two graphs G and H is the graph obtained by taking one copy of G and n copies of H and if $V(G \star H) = \{v_{1_1}, \dots, v_{1_m}, v_{2_1}, \dots, v_{2_m}, \dots, v_{n_1}, \dots, v_{n_m}, v_1, \dots, v_n\}$, then for each $i \in \{1, 2, \dots, n\}$,

every neighbor of v_i is adjacent to v_{i_k} for each $k \in \{1, 2, \dots, m\}$, again v_{i_k} is considered the same as u_k in the i^{th} copy of H . Let σ be a permutation of n items $\{1, 2, \dots, n\}$ which is written in the one-line notation as $\sigma(1), \sigma(2), \dots, \sigma(n)$, that is, an ordered arrangement of $\{1, 2, \dots, n\}$. If for each $i \in \{1, 2, \dots, n\}$, $\sigma(i) = i$, we call σ the identity permutation. Now, motivated by [1] and the concept of permutation, we define the identity-extended corona and identity-extended neighborhood corona as follows:

Definition 2.1. The identity-extended corona $I_{ex}(G \circ H)$ of two graphs G and H is the graph obtained by taking the corona $G \circ H$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertex v_{j_k} of j^{th} copy of H , provided the vertices v_i and v_j are adjacent in G ; see Figure 1.

FIGURE 1. the corona and the identity-extended corona of two graphs P_3 and C_3

Definition 2.2. The identity-extended neighborhood corona $I_{ex}(G \star H)$ of two graphs G and H is the graph obtained by taking the neighborhood corona $G \star H$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertex v_{j_k} of j^{th} copy of H , provided the vertices v_i and v_j are adjacent in G ; see Figure 2.

The following definitions are two other extensions of corona and neighborhood corona of graphs.

Definition 2.3. The neighborhood extended corona $N_{ex}(G \circ H)$ of two graphs G and H is the graph obtained by taking the corona $G \circ H$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertices v_{j_l} of j^{th} copy of H , provided the vertices v_i and v_j are adjacent in G and u_k and u_l are adjacent in H ; see Figure 3.

Definition 2.4. The neighborhood extended neighborhood corona $N_{ex}(G \star H)$ of two graphs G and H is the graph obtained by taking the neighborhood corona $G \star H$ and joining the vertex v_{i_k} of i^{th} copy of H to the vertices v_{j_l} of j^{th} copy of H , provided the vertices v_i and v_j are adjacent in G and u_k and u_l are adjacent in H ; see Figure 3.

FIGURE 2. the neighborhood corona and the identity-extended neighborhood corona of two graphs P_3 and C_3

FIGURE 3. Two neighborhood extended corona of two graphs P_3 and C_3

In this paper, suppose that $J = [e_{ij}]_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to one and $J'_{s \times t}$ obtained by replacing every entry of $J_{s \times t}$ by 0 except e_{11} . The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{n \times m}$ and $B = (b_{ij})_{p \times q}$ is the $np \times mq$ matrix obtained from A by replacing each entry a_{ij} by $a_{ij}B$, see [14]. For matrices A, B, C and D , $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist.

The M-coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1} \mathbf{1}_n,$$

where $\mathbf{1}_n$ denotes the column vector of dimension n with all the components equal to one; see [22].

Proposition 2.5 ([8]). *Let M be an $n \times n$ matrix with each row sum equal to constant c , then*

$$\Gamma_M(x) = \frac{n}{x - c}.$$

Proposition 2.6 ([20]). *Let A be an $n \times n$ real matrix, then*

$$\det(xI_n - A - \alpha J_{n \times n}) = (1 - \alpha \Gamma_A(x)) \cdot \det(xI_n - A).$$

Proposition 2.7 ([25]). *Let M_1, M_2, M_3 and M_4 be respectively $p \times p, p \times q, q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible. Then*

$$\begin{aligned} \det\left(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}\right) &= \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3) \\ &= \det(M_1) \cdot \det(M_4 - M_3 M_1^{-1} M_2), \end{aligned}$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of M_4 and M_1 , respectively.

In this paper, we apply the Laplace's expansion of a determinants by complementary minors several times; For more details, one can see page 36 of [11].

3. SPECTRA OF THE IDENTITY-EXTENDED CORONA AND IDENTITY-EXTENDED NEIGHBORHOOD CORONA

In this section, G is an arbitrary graph with n vertices and H is a r -regular graph with m vertices. To write the adjacency matrix of the identity-extended corona and identity-extended neighborhood corona, we maintain the arrangement of vertices in $V(G \circ H)$. First, we present the spectra of $I_{ex}(G \circ H)$.

Theorem 3.1. *The adjacency spectrum of $I_{ex}(G \circ H)$ is:*

- i) $\lambda_i(H) + \lambda_j(G)$ for $i = 2, \dots, m$ and $j = 1, \dots, n$;
- ii) $(2\lambda_j(G) + r \pm \sqrt{r^2 + 4m})/2$ for $j = 1, \dots, n$.

Proof. The adjacency matrix of $I_{ex}(G \circ H)$ can be expressed in the following form:

$$A(I_{ex}(G \circ H)) = \begin{bmatrix} I_n \otimes A(H) + A(G) \otimes I_m & I_n \otimes J_{m \times 1} \\ I_n \otimes J_{1 \times m} & A(G) \end{bmatrix}.$$

$A(H)$ is a real Hermitian matrix, so $A(H)$ is orthogonally diagonalizable and Since H is r -regular, $A(H) = PD(H)P^T$ where $D(H) = \text{diag}(r, \lambda_2(H), \dots, \lambda_m(H))$ and P is a square matrix

of order m such that its first column vector is $\frac{1}{\sqrt{m}}(1, 1, \dots, 1)$ and $P^T P = I_m$. Therefore,

$$\begin{aligned} A(I_{ex}(G \circ H)) &= \begin{bmatrix} I_n \otimes PD(H)P^T + A(G) \otimes I_m & I_n \otimes J_{m \times 1} \\ I_n \otimes J_{1 \times m} & A(G) \end{bmatrix} \\ &= \begin{bmatrix} I_n \otimes P & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n \otimes D(H) + A(G) \otimes I_m & I_n \otimes P^T J_{m \times 1} \\ I_n \otimes J_{1 \times m} P & A(G) \end{bmatrix} \begin{bmatrix} I_n \otimes P^T & 0 \\ 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} I_n \otimes P & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n \otimes D(H) + A(G) \otimes I_m & I_n \otimes \sqrt{m} J'_{m \times 1} \\ I_n \otimes \sqrt{m} J'_{1 \times m} & A(G) \end{bmatrix} \begin{bmatrix} I_n \otimes P^T & 0 \\ 0 & I_n \end{bmatrix}. \end{aligned}$$

So $A(I_{ex}(G \circ H))$ is similar to

$$B_0 = \begin{bmatrix} I_n \otimes D(H) + A(G) \otimes I_m & I_n \otimes \sqrt{m} J'_{m \times 1} \\ I_n \otimes \sqrt{m} J'_{1 \times m} & A(G) \end{bmatrix},$$

and

$$\begin{aligned} \det(xI - A(I_{ex}(G \circ H))) &= \det(xI - B_0) \\ &= \det \begin{bmatrix} I_n \otimes (xI_m - D(H)) - A(G) \otimes I_m & -I_n \otimes \sqrt{m} J'_{m \times 1} \\ -I_n \otimes \sqrt{m} J'_{1 \times m} & xI_n - A(G) \end{bmatrix}. \end{aligned}$$

Using the Laplace's expansion of $\det(xI - B_0)$ by $(mi + 2), (mi + 3), \dots, (mi + m)^{th}$ columns for $i = 0, 1, \dots, n - 1$, we see that $\det(xI - B_0) = M.M_1$ where $M = \det(B_1)$ and $B_1 = I_n \otimes \text{diag}(x - \lambda_2(H), \dots, x - \lambda_m(H)) - (A(G) \otimes I_{m-1})$, and also

$$M_1 = \det \begin{bmatrix} (x - r)I_n - A(G) & -\sqrt{m}I_n \\ -\sqrt{m}I_n & xI_n - A(G) \end{bmatrix}$$

is the complementary minor of M . Again expanding $\det(B_1)$ by Laplace's method in the similar way and continuing this method $(m - 2)$ times, we have

$$\begin{aligned} \det(B_1) &= \prod_{i=2}^m \det((x - \lambda_i(H))I_n - A(G)) \\ &= \prod_{i=2}^m \prod_{j=1}^n (x - \lambda_i(H) - \lambda_j(G)) \\ &= \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - (\lambda_i(H) + \lambda_j(G))). \end{aligned}$$

Again $A(G)$ is orthogonally diagonalizable, so we can easily see that

$$M_1 = \det \begin{bmatrix} (x - r)I_n - D(G) & -\sqrt{m}I_n \\ -\sqrt{m}I_n & xI_n - D(G) \end{bmatrix}.$$

By Proposition 2.7,

$$\begin{aligned}
 M_1 &= \det(xI_n - D(G)) \cdot \det((x - r)I_n - D(G) - m(xI_n - D(G))^{-1}) \\
 &= \det(xI_n - D(G)) \cdot \prod_{i=1}^n \left(x - r - \lambda_i(G) - \frac{m}{x - \lambda_i(G)}\right) \\
 &= \prod_{i=1}^n (x - \lambda_i(G)) \cdot \prod_{i=1}^n \frac{x^2 - (2\lambda_i(G) + r)x + \lambda_i^2(G) + r\lambda_i(G) - m}{x - \lambda_i(G)} \\
 &= \prod_{i=1}^n (x^2 - (2\lambda_i(G) + r)x + (\lambda_i^2(G) + r\lambda_i(G) - m)).
 \end{aligned}$$

Finally,

$$\begin{aligned}
 \det(xI - A(I_{ex}(G \star H))) &= \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - (\lambda_i(H) + \lambda_j(G))) \\
 &\quad \cdot \prod_{i=1}^n (x^2 - (2\lambda_i(G) + r)x + (\lambda_i^2(G) + r\lambda_i(G) - m)).
 \end{aligned}$$

□

Now, we present the spectra of the identity-extended neighborhood corona $I_{ex}(G \star H)$ of two graphs G and H , where H is regular.

Theorem 3.2. *The adjacency spectrum of $I_{ex}(G \star H)$ is as follows:*

- i) $\lambda_i(H) + \lambda_j(G)$ for $i = 2, \dots, m$ and $j = 1, \dots, n$;
- ii) $(2\lambda_j(G) + r \pm \sqrt{r^2 + 4m\lambda_j^2(G)})/2$ for $j = 1, \dots, n$.

Proof. The adjacency matrix of $I_{ex}(G \star H)$ can be written as:

$$A(I_{ex}(G \star H)) = \begin{bmatrix} I_n \otimes A(H) + A(G) \otimes I_m & A(G) \otimes J_{m \times 1} \\ A(G) \otimes J_{1 \times m} & A(G) \end{bmatrix}.$$

Similar to the proof of Theorem 3.1, $A(I_{ex}(G \star H))$ is similar to

$$B_0 = \begin{bmatrix} I_n \otimes D(H) + A(G) \otimes I_m & A(G) \otimes \sqrt{m}J'_{m \times 1} \\ A(G) \otimes \sqrt{m}J'_{1 \times m} & A(G) \end{bmatrix},$$

and

$$\begin{aligned}
 \det(xI - A(I_{ex}(G \star H))) &= \det(xI - B_0) \\
 &= \det \begin{bmatrix} I_n \otimes (xI_m - D(H)) - A(G) \otimes I_m & -A(G) \otimes \sqrt{m}J'_{m \times 1} \\ -A(G) \otimes \sqrt{m}J'_{1 \times m} & xI_n - A(G) \end{bmatrix}.
 \end{aligned}$$

Again similar to the proof of Theorem 3.1; using the Laplace's expansion,

$$\det(xI - A(I_{ex}(G \star H))) = \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - (\lambda_i(H) + \lambda_j(G))) \\ \cdot \det \begin{bmatrix} (x-r)I_n - A(G) & -\sqrt{m}A(G) \\ -\sqrt{m}A(G) & xI_n - A(G) \end{bmatrix}.$$

$A(G)$ is orthogonally diagonalizable, so we can easily see that

$$\det(xI - A(I_{ex}(G \star H))) = \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - (\lambda_i(H) + \lambda_j(G))) \\ \cdot \det \begin{bmatrix} (x-r)I_n - D(G) & -\sqrt{m}D(G) \\ -\sqrt{m}D(G) & xI_n - D(G) \end{bmatrix} \\ = \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - (\lambda_i(H) + \lambda_j(G))) \\ \cdot \prod_{i=1}^n (x^2 - (2\lambda_i(G) + r)x + r\lambda_i(G) + (1-m)\lambda_i^2(G)).$$

□

4. SPECTRA OF THE NEIGHBORHOOD EXTENDED CORONA AND NEIGHBORHOOD EXTENDED NEIGHBORHOOD CORONA

In this section, we present the spectra of the neighborhood extended corona $N_{ex}(G \circ H)$ and the neighborhood extended neighborhood corona $N_{ex}(G \star H)$ of two graphs G and H where G is an arbitrary n -vertices and H is a r -regular m -vertices graph. To write the adjacency matrix of these two operations, we maintain the arrangement of vertices in $V(G \star H)$.

Theorem 4.1. *The adjacency spectrum of $N_{ex}(G \circ H)$ is:*

- i) $\lambda_i(H)(1 + \lambda_j(G))$ for $i = 2, \dots, m$ and $j = 1, \dots, n$;
- ii) $((r+1)\lambda_j(G) + r \pm \sqrt{((r-1)\lambda_j(G) + r)^2 + 4m})/2$ for $j = 1, \dots, n$.

Proof. The adjacency matrix of $N_{ex}(G \circ H)$ can be expressed in the following form:

$$A(N_{ex}(G \circ H)) = \begin{bmatrix} (I_n + A(G)) \otimes A(H) & I_n \otimes J_{m \times 1} \\ I_n \otimes J_{1 \times m} & A(G) \end{bmatrix}.$$

$A(H)$ is a real Hermitian matrix and H is r -regular, so we have $A(H) = PD(H)P^T$ where $A(H) = \text{diag}(r, \lambda_2(H), \dots, \lambda_m(H))$ and P is a square matrix of order m such that its first

column vector is $\frac{1}{\sqrt{m}}(1, 1, \dots, 1)$ and $P^T P = I_m$. Therefore,

$$A(N_{ex}(G \circ H)) = \begin{bmatrix} I_n \otimes P & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} (I_n + A(G)) \otimes D(H) & I_n \otimes \sqrt{m} J'_{m \times 1} \\ I_n \otimes \sqrt{m} J'_{1 \times m} & A(G) \end{bmatrix} \begin{bmatrix} I_n \otimes P^T & 0 \\ 0 & I_n \end{bmatrix}.$$

So $A(N_{ex}(G \circ H))$ is similar to

$$B_0 = \begin{bmatrix} (I_n + A(G)) \otimes D(H) & I_n \otimes \sqrt{m} J'_{m \times 1} \\ I_n \otimes \sqrt{m} J'_{1 \times m} & A(G) \end{bmatrix},$$

and

$$\det(xI - A(N_{ex}(G \circ H))) = \det(xI - B_0).$$

Now, use the Laplace's expansion of $\det(xI - B_0)$ by $(mi + 2), (mi + 3), \dots, (mi + m)^{th}$ columns for $i = 0, 1, \dots, n - 1$. We have

$$\det(xI - B_0) = \det(B_1) \cdot M_1,$$

where $B_1 = I_n \otimes \text{diag}(x - \lambda_2(H), \dots, x - \lambda_m(H)) - (A(G) \otimes \text{diag}(\lambda_2(H), \dots, \lambda_m(H)))$ and

$$M_1 = \det \begin{bmatrix} (x - r)I_n - rA(G) & -\sqrt{m}I_n \\ -\sqrt{m}I_n & xI_n - A(G) \end{bmatrix}$$

is the complementary minor of $\det(B_1)$. Again expanding $\det(B_1)$ by Laplace's method in the similar way and continuing this method $(m - 2)$ times, we have

$$\begin{aligned} \det(B_1) &= \prod_{i=2}^m \det((x - \lambda_i(H))I_n - \lambda_i(H)A(G)) \\ &= \prod_{\substack{2 \leq i \leq m \\ 1 \leq j \leq n}} (x - \lambda_i(H)(1 + \lambda_j(G))). \end{aligned}$$

$A(G)$ is orthogonally diagonalizable, so we can easily see that

$$M_1 = \det \begin{bmatrix} (x - r)I_n - rD(G) & -\sqrt{m}I_n \\ -\sqrt{m}I_n & xI_n - D(G) \end{bmatrix}.$$

By Proposition 2.7,

$$\begin{aligned}
 M_1 &= \det(xI_n - D(G)) \cdot \det((x-r)I_n - rD(G) - m(xI_n - D(G))^{-1}) \\
 &= \det(xI_n - D(G)) \cdot \prod_{i=1}^n \left(x - r - r\lambda_i(G) - \frac{m}{x - \lambda_i(G)} \right) \\
 &= \prod_{i=1}^n (x - \lambda_i(G)) \cdot \prod_{i=1}^n \frac{x^2 - ((r+1)\lambda_i(G) + r)x + r\lambda_i^2(G) + r\lambda_i(G) - m}{x - \lambda_i(G)} \\
 &= \prod_{i=1}^n (x^2 - ((r+1)\lambda_i(G) + r)x + (r\lambda_i^2(G) + r\lambda_i(G) - m)).
 \end{aligned}$$

□

The proof of the next theorem is omitted because of similarity.

Theorem 4.2. *The adjacency spectrum of $N_{ex}(G \star H)$ is:*

- i) $\lambda_i(H)(1 + \lambda_j(G))$ for $i = 2, \dots, m$ and $j = 1, \dots, n$;
- ii) $((r+1)\lambda_j(G) + r \pm \sqrt{((r-1)\lambda_j(G) + r)^2 + 4m\lambda_j^2(G)})/2$ for $j = 1, \dots, n$.

5. NEW CLASSES OF INTEGRAL GRAPHS

In this section, we construct infinitely many families of integral graphs. The next theorem follows from Theorem 3.1, 3.2, 4.1 and 4.2.

Theorem 5.1. *suppose that G is an integral graph with n vertices and H is a r -regular integral graph with m vertices, then*

- (1) *The graph $I_{ex}(G \circ H)$ is integral if $m = h(h \pm r)$ for some positive integer h .*
- (2) *The graph $I_{ex}(G \star H)$ is integral if for every $j = 1, 2, \dots, n$, $m\lambda_j^2(G) = h(h \pm r)$ for some positive integer h .*
- (3) *$N_{ex}(G \circ pK_2)$ is integral graph if $8p + 1$ is perfect square.*
- (4) *$N_{ex}(G \star pK_1)$ is integral graph if $4p + 1$ is perfect square.*

As an special case of previous theorem, we have the following corollary.

Corollary 5.2. *suppose that G is an integral graph, then for each positive integer p and q ,*

- (1) *$I_{ex}(G \circ K_p)$, $I_{ex}(G \circ pK_{p,p})$ and $I_{ex}(G \circ C_3)$ are integral graphs.*
- (2) *$I_{ex}(qK_2 \star K_p)$ and $I_{ex}(qK_2 \star pK_{p,p})$ are integral graphs.*

Remark 5.3. In Theorem 5.1 and Corollary 5.2, one can replace G with any graph of the classes of integral graphs has obtained so far. in fact, all the integral graphs obtained from new operations on the graphs can replace by G ; for example, one can see the class of integral graphs obtained in [1, 2, 16, 17].

6. QUESTION

Regarding the concept of permutation and definition of identity-extended corona and identity-extended neighborhood corona, it seems that this concept can be generalized to the σ -extended corona and σ -extended neighborhood corona. It seems that such an extension of the corona corresponds to many molecular bonds, and thus by obtaining a variety of spectra of such expansion, some of the indices associated with those bonds would be calculated. But this question still remains open, how can obtain these spectrums?

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