



ON DUAL OF THE GENERALIZED SPLITTING MATROIDS

GH. GHAFARI*, GH. AZADI, H. AZANCHILER

Communicated by S. Alikhani

ABSTRACT. Given a binary matroid M and a subset $T \subseteq E(M)$, Luis A. Goddyn posed a problem that the dual of the splitting of M , i.e., $((M_T)^*)$ is not always equal to the splitting of the dual of M , $((M^*)_T)$. This persuaded us to ask if we can characterize those binary matroids for which $(M_T)^* = (M^*)_T$. Santosh B. Dhotre answered this question for a two-element subset T . In this paper, we generalize his result for any subset $T \subseteq E(M)$ and exhibit a criterion for a binary matroid M and subsets T for which $(M_T)^*$ and $(M^*)_T$ are the equal. We also show that there is no subset $T \subseteq E(M)$ for which, the dual of element splitting of M , i.e., $((M'_T)^*)$ equals to the element splitting of the dual of M , $((M^*)'_T)$.

1. INTRODUCTION

Fleischner [2] defined the splitting operation on graphs as follows: Let G be a connected graph and v be a vertex of degree at least three in G . If $x = vv_1$ and $y = vv_2$ are two edges incident at v , then the splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y , and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2 . The transition from G to $G_{x,y}$ is called the splitting operation on G .

MSC(2010): Primary: 05B35

Keywords: Binary matroid, dual of a matroid, n-connected matroid, splitting operation, cocircuit.

Received: 21 May 2018, Accepted: 19 October 2018

*Corresponding author

Raghunathan et al. [4] extended the notion of the splitting operation from graphs to binary matroids for every pair x, y of $E(M)$. M. M. Shikare and G. Azadi [6] generalized this operation for any subset $T \subseteq E(M)$ for binary matroids as follows: Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ representing the matroid M . Let T be a subset of E and A_T be the matrix that is obtained by adding an extra row to A in which the row being zero everywhere except for the columns corresponding to T , where it takes the value 1. Let M_T be the matroid represented by the matrix A_T , we say that M_T has been obtained from M by splitting the set T .

Slater [7] defined the n -point-splitting operation on graphs as follows: Let G be a graph and u be a vertex of G such that $\deg(u) \geq 2n - 2$ in which $n \in \mathbb{N}$. Let H be the graph obtained from G by replacing u by two adjacent vertices u_1 and u_2 , if a vertex v is adjacent to u in G , written $v \text{ adj } u$, then make $v \text{ adj } u_1$ or $\text{adj } u_2$ (but not both) such that $\deg(u_1) \geq n$ and $\deg(u_2) \geq n$. We call the transition from G to H a n -point-splitting operation.

G. Azadi [6] extended the notation of n -point-splitting operation from graphs to binary matroid as follows. Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ representing M . Let T be a subset of E , and A'_T be the matrix obtained by adjoining an extra row to A in which the row being zero everywhere except for the columns corresponding to the elements of T where it takes the value 1, and adjoining an extra column (corresponding to a , $a \notin E(M)$) with this column being zero everywhere except in the last row where it takes the value 1. Let M'_T be the matroid represented by the matrix A'_T , we say that M'_T has been obtained from M by the element splitting of the set T .

Let M be the matroid (E, \mathcal{I}) and suppose that $T \subseteq E$. Let $\mathcal{I}|T$ be $\{I \subseteq T : I \in \mathcal{I}\}$. Then it is easy to see that the pair $(T, \mathcal{I}|T)$ is a matroid. We call this matroid the restriction of M to T , which is obtained by deleting $E - T$ from M . It is denoted by $M|T$ or $M \setminus (E - T)$. Suppose e is an element of E and let $M/e = (M^* \setminus e)^*$, in which M^* is dual of M . We shall call M/e , the contraction of M onto $E - \{e\}$ or the contraction of e from M . For $X \subseteq E$, let

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

We call λ_M the connectivity function of M . Let k be a positive integer. A k -separation of M is a partition $\{X, Y\}$ of $E(M)$ such that $\min\{|X|, |Y|\} \geq k$, and $\lambda_M(X) \leq k - 1$. For all $n \geq 2$, M is n -connected if, for all k in $\{1, 2, \dots, n - 1\}$, M has no k -separation.

Lemma 1.1. [3] *Let M be a matroid with ground set E . If $X \subseteq E$, then*

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

Various properties of the splitting matroids are explored in [4] and [6]. For the standard terminology in matroid we refer to [3].

2. DUAL OF THE BINARY SPLITTING MATROID

In this section, we consider the problem of finding a necessary and sufficient condition for a matroid M and a subset T of $E(M)$ for which $(M^*)_T = (M_T)^*$. As pointed above, the dual of the splitting of a matroid M is not always equal to the splitting of its dual. The following proposition is necessary in our discussion.

Proposition 2.1. [3] *Let A be a binary representation of a rank- r binary matroid M . Then the cocircuit space of M equals to the row space of A . Moreover, this space has dimension r and is the orthogonal subspace of the circuit space of M .*

The following lemma is an immediate consequence of the Proposition (1)

Lemma 2.2. *Let M be a binary matroid and $T \subseteq E(M)$. Then $M_T = M$ if and only if T is a union of the disjoint cocircuits of M .*

Corollary 2.3. *Let M be a binary matroid and $T \subseteq E(M)$. If T is not a union of the disjoint cocircuits of M , then $M_T \neq M$ and $r'(M_T) = r(M) + 1$, where r' and r are the rank functions of M_T and M , respectively.*

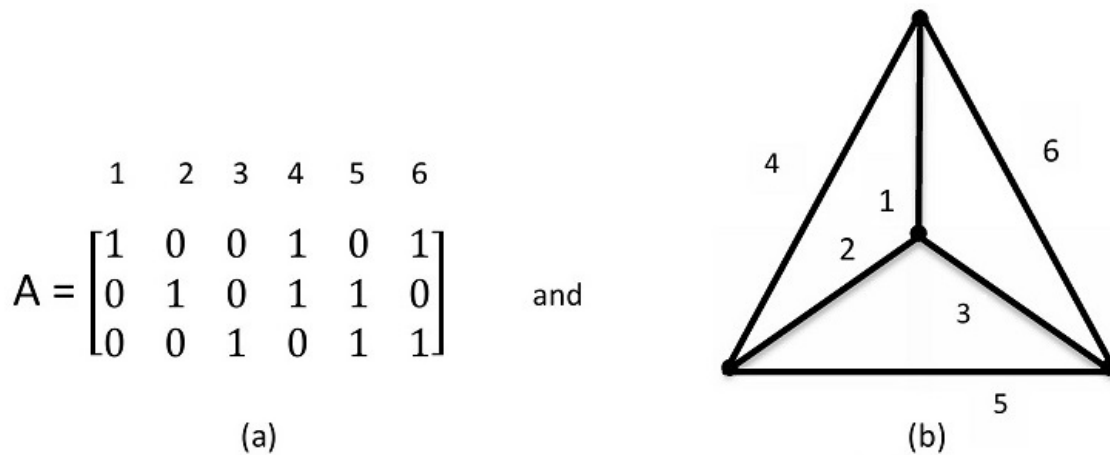


FIGURE 1

Remark 2.4. Consider the vector matroid M of the matrix A over $GF(2)$, that is a representation of $M(K_4)$ (see Figure 1), and take $T = \{3, 5, 6\}$. Since T is a cocircuit of $M(K_4)$, so by Proposition (1), $A_T = A$, therefore $(M_T)^* = M^*(K_4)$, (see Figure 1(a) and 2(a)).

Moreover by the following representation for $(M^*)_T$ (see Figure 2(b)); We conclude that $(M^*)_T$ is different from $(M_T)^*$.

$$\begin{array}{ccc}
 \begin{array}{cccccc}
 1 & 2 & 3 & 4 & 5 & 6 \\
 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} & & \text{and} & & \begin{array}{cccccc}
 4 & 5 & 6 & 1 & 2 & 3 \\
 \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \\
 \text{(a)} & & & & \text{(b)}
 \end{array}
 \end{array}$$

FIGURE 2

Luis A. Goddyn asked the question: Find the condition for M and x, y such that the two matroids $(M_{x,y})^*$ and $(M^*)_{x,y}$ will be the same. Santosh B. Dhotre answered this question by the following theorem.

Theorem 2.5. [1] *Let M be a binary matroid, and $x, y \in E(M)$. Then $(M^*)_{x,y} = (M_{x,y})^*$ if and only if x and y are in series and $\{x, y\}$ forms a circuit of M .*

In this paper, we generalize this theorem for any subset $T \subseteq E(M)$ and exhibit a criterion for a binary matroid M and subset T for which $(M_T)^*$ and $(M^*)_T$ are the same.

Definition 2.6. We call a matroid M an Eulerian matroid if there exist disjoint circuits C_1, C_2, \dots, C_p such that $E(M) = C_1 \cup C_2 \cup \dots \cup C_p$.

Definition 2.7. We call a matroid M bipartite, if every circuit of M has even cardinality.

Proposition 2.8. [8] *Let M be a binary matroid. Then M is Eulerian if and only if M^* is bipartite.*

To prove one of our main results, we need the following theorem.

Theorem 2.9. *Let M be a binary matroid and $T \subseteq E(M)$. Then $(M^*)_T = (M_T)^*$ if and only if T is a union of the disjoint cocircuits and also a union of the disjoint circuits of M .*

Proof. Suppose that T is a union of the disjoint cocircuits and a union of the disjoint circuits of M . Then by Lemma (2) and its dual, $(M_T)^* = (M^*)_T$. So the result holds.

Conversely, suppose that $(M_T)^* = (M^*)_T$. We show that T is a union of the disjoint cocircuits and also a union of the disjoint circuits of M . Suppose this is not true. We consider the following cases:

Case (i): If T is neither a union of the disjoint cocircuits nor a union of the disjoint circuits of M . Then by Corollary (1), $r(M_T) = r(M) + 1$. So

$$\begin{aligned}
 r((M_T)^*) &= |E(M)| - (r(M) + 1) \\
 &= |E(M)| - r(M) - 1 \\
 (1) \qquad &= r(M^*) - 1
 \end{aligned}$$

On the other hand, by the dual of Corollary (1),

$$(2) \qquad r((M^*)_T) = r(M^*) + 1$$

But, $|E(M)| - r(M) = r(M^*)$. Since $(M_T)^* = (M^*)_T$ so we have $r(M_T)^* = r(M^*)_T$; On the other hand by (1) and (2), we deduce that, $r(M^*) + 1 = r(M^*) - 1$, a contradiction.

Case (ii): If T is a union of the disjoint cocircuits but not a union of the disjoint circuits of M , then we have $(M_T)^* = M^*$, so

$$\begin{aligned}
 r((M_T)^*) &= r(M^*) \\
 &= |E(M)| - r(M) \\
 (3) \qquad &= r(M^*)
 \end{aligned}$$

On the other hand, by Lemma (2) along with the Corollary (1) applied to dual, we have

$$(4) \qquad r((M^*)_T) = r(M^*) + 1$$

But $(M_T)^* = (M^*)_T$, therefore $r((M_T)^*) = r((M^*)_T)$. On the other hand by (3) and (4), we deduce that, $r(M^*) = r(M^*) + 1$, a contradiction.

Case (iii): If T is a union of the disjoint circuits but not the disjoint union of cocircuits of M , then by similar argument in case 2 on the dual of M , we get a contradiction, so the result holds. This completes the proof. □

Corollary 2.10. *Suppose M is a binary matroid on E and $T \subseteq E(M)$. Then $M|T$ is Eulerian and bipartite if and only if $(M^*)_T = (M_T)^*$.*

Proof. It is an immediate cosequence of the Theorem (2) and the Proposition (2). □

Corollary 2.11. *Let M be a n -connected binary matroid with at least $(2n - 1)$ elements. Then for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* \neq (M^*)_T$.*

Proof. Suppose that there is a subset T with $|T| = n$ in which $(M_T)^* = (M^*)_T$. Then by Theorem (2), T is a union of the disjoint circuits and union of the disjoint cocircuits of M . So

$r(T) \leq n - k$ and $r^*(T) \leq n - j$ for some $j, k \geq 1$. On the other hand we have

$$\begin{aligned}\lambda_M(T) &= r(T) + r^*(T) - |T| \\ &= n - k + n - j - n \\ &\leq n - k - j.\end{aligned}$$

And since M has at least $(2n-1)$ elements, we conclude that $(T, E(M)-T)$ is a $(n-k-j+1)$ -separation for M , a contradiction. \square

Definition 2.12. Let F be an arbitrary field and $V=(v_1, v_2, \dots, v_n)$ be a member of $V(n, F)$. The support of V is $\{i : v_i \neq 0\}$.

Proposition 2.13. [3] *Let M be a binary matroid on E . Then $E(M)$ can be partitioned into circuits if and only if there is a basis of the cocircuit space all of whose member have even support.*

Corollary 2.14. *Let M a binary matroid on E . Then M is eulerian if and only if there is a matrix A representing M and has even non-zero entries in each rows.*

Proposition 2.15. [3] *Let C be a circuit of a binary matroid M and e be an element of $E(M) - C$. Then, in M/e , either C is a circuit or C is a disjoint union of two circuits. In both case, M/e has no other circuits contained in C .*

The following corollary is an immediate consequence of Proposition (4).

Corollary 2.16. *Let M be binary matroid on E , $T \subsetneq E$ and $y \in E - T$. If T be a union of the disjoint cocircuits of M , then T is also a union of the disjoint cocircuits of $M|(T \cup \{y\})$.*

Theorem 2.17. *Let M be a binary matroid on E . Then for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* = (M^*)_T$ if and only if $n = |E(M)|$ and M is Eulerian and bipartite. Moreover, $n = 2k$ for some $k \in \mathbb{N}$, that is, n is even.*

Proof. Since M is Eulerian, bipartite and $T = E(M)$, so by Theorem (2), $(M_T)^* = (M^*)_T$ and the result holds.

Conversely, suppose for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* = (M^*)_T$. We prove that $T = E(M)$. Suppose $T \neq E(M)$ and $y \in E(M) - T$. Since $(M_T)^* = (M^*)_T$, so by Theorem (2) T is a union of the disjoint circuits and also a union of the disjoint cocircuits of M . Let A be a matrix that representing M , then by Corollary (4), in $M[A]|T$ the number of non-zero entries in each row is even. Suppose that $T \cup \{y\}$ is in the first columns of A .

set $T' = (T - \{x_1\}) \cup \{y\}$, where $x_1 \in T$. Now T' is an n -element subset of $E(M)$. But y must be parallel to x_1 , otherwise in some row of $M[A]|T'$ in which y has non-zero entries, we

have odd non-zero entries, a contradiction. Now there is an element $x_1 \neq x_i \in T$ which is not parallel to x_1 and y , otherwise $M|(T \cup \{y\}) \cong U_{1,|T|+1}$, this contradicts the Corollary (5).

Let $T'' = \{T \cup \{y\}\} - \{x_i\}$, where $x_1 \neq x_i \in T$. Now T'' is an n -element set of $E(M)$ in which $M|T''$ is not Eulerian, a contradiction. So T must be equal to $E(M)$. Now Theorem (2) and Proposition (2) show that M is Eulerian and bipartite.

Moreover since $E(M)$ is a union of the disjoint circuits, also a union of the disjoint cocircuits of M , and the fact that the intersection of circuits and cocircuits of a binary matroid has even element, we conclude that n is even. □

3. ON DUAL OF THE BINARY ELEMENT SPLITTING MATROID

In this section, we show that $(M^*)'_T \neq (M'_T)^*$ for every $T \subseteq E(M)$ in the element splitting. As we pointed out in abstract, in this section we prove that for every $T \subseteq E(M)$, $(M^*)'_T \neq (M'_T)^*$.

Lemma 3.1. [6] *Let M be a binary matroid on E . Then for any $X \subseteq E(M)$, $r'(X \cup \{a\}) = r(X) + 1$ where r and r' denote the rank functions of M and M'_T , respectively.*

Corollary 3.2. *If M is a binary matroid on E and $T \subseteq E(M)$, then*

$$r'(M'_T) = r(M) + 1.$$

where r and r' denote the rank functions of M and M'_T , respectively.

Theorem 3.3. *Let M be a binary matroid on E and $T \subseteq E(M)$. Then for every $T \subseteq E(M)$, $(M^*)'_T \neq (M'_T)^*$.*

Proof. Suppose that there is a $T \subseteq E(M)$, such that $(M^*)'_T = (M'_T)^*$. Let r be the rank function. Then

$$(5) \quad r((M^*)'_T) = r((M'_T)^*).$$

But, by Corollary (6), $r(M'_T) = r(M) + 1$, so

$$\begin{aligned} r((M'_T)^*) &= |E(M) \cup \{a\}| - (r(M) + 1) \\ &= |E(M)| - r(M) \\ (6) \quad &= r(M^*). \end{aligned}$$

Again by Corollary (6),

$$(7) \quad r((M^*)'_T) = r(M^*) + 1.$$

Therefore by (5), (6) and (7) we have $r(M^*) + 1 = r(M^*)$, a contradiction. This completes the proof. \square

REFERENCES

- [1] S. B. Dhotre, *A note on the dual of the splitting matroid*, Lobachevskii J. Math., **33**, (2012), 229-231.
- [2] H. Fleischner, *Eulerian Graphs and Related Topics*, North Holland, Amsterdam, (1990).
- [3] J. G. Oxley, *Matroid Theory*, Oxford university press, New York, (2011).
- [4] T. T. Raghunathan, M. M. Shikare and B. N. Waphare, *Splitting in a binary matroid*, Discrete math., **184**, (1998), 267-271.
- [5] M. M. Shikare, Gh. Azadi, *Determination of the bases of a splitting matroid*, European J. combin., **24**, (2003), 45-52.
- [6] M. M. Shikare, Gh. Azadi, B. N. Waphare, *Generalized splitting operation and its application*, J. Indian Math. Soc., **78**, (2011), 145-154.
- [7] P. J. Slater, *A classification of 4-connected graphs*, J. Combin. Theory, **17**, (1974), 281-298.
- [8] D. J. A. Welsh, *Eulerian and bipartite matroids*, J. Comb. Theory, **6**, (1969), 375-377.

Gh. Ghafari

Department of Mathematics
Urmia University, Urmia, Iran
gh.ghafari@urmia.ac.ir

Gh. Azadi

Department of Mathematics
Urmia University, Urmia, Iran
gh.azadi@urmia.ac.ir

H. Azanchiler

Department of Mathematics
Urmia University, Urmia, Iran
h.azanchiler@urmia.ac.ir