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ON DUAL OF THE GENERALIZED SPLITTING MATROIDS

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Abstract. Given a binary matroid M and a subset $T \subseteq E(M)$, Luis A. Goddyn posed a problem that the dual of the splitting of M, i.e., $((M_T)^*)$ is not always equal to the splitting of the dual of M, $((M^*)_T)$. This persuade us to ask if we can characterize those binary matroids for which $(M_T)^* = (M^*)_T$. Santosh B. Dhotre answered this question for a two-element subset T. In this paper, we generalize his result for any subset $T \subseteq E(M)$ and exhibit a criterion for a binary matroid M and subsets T for which $(M_T)^*$ and $(M^*)_T$ are the equal. We also show that there is no subset $T \subseteq E(M)$ for which, the dual of element splitting of M, i.e., $((M_T^{'})^*)$ equals to the element splitting of the dual of M, $((M^*)_T^{'})$.

1. Introduction

Fleischner [2] defined the splitting operation on graphs as follows: Let G be a connected graph and v be a vertex of degree at least three in G. If $x = vv_1$ and $y = vv_2$ are two edges incident at v, then the splitting away the pair x, y from v results in a new gragh $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2. The transition from G to $G_{x,y}$ is called the splitting operation on G.

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Raghunathan et al. [4] extended the notion of the splitting operation from graphs to binary matroids for every pair x, y of $E(M)$. M. M. Shikare and G. Azadi [6] generalized this operation for any subset $T \subseteq E(M)$ for binary matroids as follows: Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ representing the matroid M. Let T be a subset of E and A_T be the matrix that is obtained by adding an extra row to A in which the row being zero everywhere except for the columns corresponding to T, where it takes the value 1. Let M_T be the matroid represented by the matrix A_T, we say that M_T has been obtained from M by splitting the set T.

Slater [7] defined the n-point-splitting operation on graphs as follows: Let G be a graph and u be a vertex of G such that $deg(u) \geq 2n-2$ in which $n \in N$. Let H be the graph obtained from G by replacing u by two adjacent vertices u_1 and u_2, if a vertex v is adjacent to u in G, written v adj u, then make v adj u_1 or adj u_2 (but not both) such that $deg(u_1) \geq n$ and $deg(u_2) \geq n$. We call the transition from G to H a n-point-splitting operation.

G. Azadi [6] extended the notation of n-point-splitting operation from graphs to binary matroid as follows. Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ representing M. Let T be a subset of E, and A'_T be the matrix obtained by adjoining an extra row to A in which the row being zero everywhere except for the columns corresponding to the elements of T where it takes the value 1, and adjoining an extra column (corresponding to a, $a \notin E(M)$) with this column being zero everywhere except in the last row where it takes the value 1. Let M'_T be the matroid represented by the matrix A'_T, we say that M'_T has been obtained from M by the element splitting of the set T.

Let M be the matroid $(E, \mathcal{I})$ and suppose that $T \subseteq E$. Let $\mathcal{I}|T$ be $\{I \subseteq T : I \in \mathcal{I}\}$. Then it is easy to see that the pair $(T, \mathcal{I}|T)$ is a matroid. We call this matroid the restriction of M to T, which is obtained by deleting $E - T$ from M. It is denoted by $M|T$ or $M\backslash(E-T)$. Suppose e is an element of E and let $M/e = (M^*\backslash e)^*$, in which M^* is dual of M. We shall call M/e, the contraction of M onto $E - \{e\}$ or the contraction of e from M. For $X \subseteq E$, let

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

We call λ_M the connectivity function of M. Let k be a positive integer. A k-separation of M is a partition $\{X, Y\}$ of E(M) such that $min\{|X|, |Y|\} \geq k$, and $\lambda_M(X) \leq k-1$. For all $n \geq 2$, M is n-connected if, for all k in $\{1, 2, ..., n-1\}$, M has no k-separation.

Lemma 1.1. [3] *Let M be a matroid with ground set E. If $X \subseteq E$, then*

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

Various properties of the splitting matroids are explored in [4] and [6]. For the standard terminology in matroid we refer to [3].

2. Dual of the binary splitting matroid

In this section, we consider the problem of finding a necessary and sufficient condition for a matroid M and a subset T of $E(M)$ for which $(M^*)_T=(M_T)^*$. As pointed above, the dual of the splitting of a matroid M is not always equal to the splitting of it's dual. The following proposition is necessary in our discussion.

Proposition 2.1. [3] *Let A be a binary representation of a rank-r binary matroid M. Then the cocircuit space of M equals to the row space of A. Moreover, this space has dimension r and is the orthogonal subspace of the circuit space of M.*

The following lemma is an immediate consequence of the Proposition 2.1

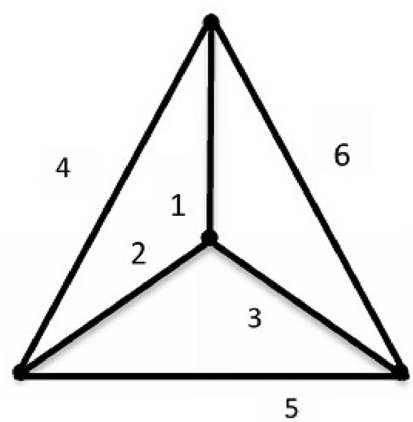
Lemma 2.2. *Let M be a binary matroid and $T \subseteq E(M)$. Then $M_T = M$ if and only if T is a union of the disjoint cocircuits of M.*

Corollary 2.3. *Let M be a binary matroid and $T \subseteq E(M)$. If T is not a union of the disjoint cocircuits of M, then $M_T \neq M$ and $r'(M_T) = r(M)+1$, where r' and r are the rank functions of M_T and M, respectively.*

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ A = & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

and

(a)



(b)

Figure 1

Remark 2.4. Consider the vector matroid M of the matrix A over $GF(2)$, that is a representation of $M(K_4)$ (see Figure 1), and take $T = \{3,5,6\}$. Since T is a cocircuit of $M(K_4)$, so by Proposition 2.1, $A_T = A$, therefore $(M_T)^* = M^*(K_4)$, (see Figure 1(a) and 2(a)).

Moreover by the following reprsentation for $(M^*)_T$ (see Figure 2(b)); We conclude that $(M^*)_T$ is different from $(M_T)^*$.

$$A_T = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad (M^*)_T = \begin{array}{c} \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(a) (b)

FIGURE 2

Luis A. Goddyn asked the question: Find the condition for M and x, y such that the two matroids $(M_{x,y})^*$ and $(M^*)_{x,y}$ will be the same. Santosh B. Dhotre answerd this question by the following theorem.

Theorem 2.5. [1] *Let M be a binary matroid, and $x, y \in E(M)$. Then $(M^*)_{x,y} = (M_{x,y})^*$ if and only if x and y are in series and $\{x, y\}$ forms a circuit of M.*

In this paper, we generalize this theorem for any subset $T \subseteq E(M)$ and exhibit a criterian for a binary matroid M and subset T for which $(M_T)^*$ and $(M^*)_T$ are the same.

Definition 2.6. We call a matroid M an Eulerian matroid if there exist disjoint circuits C_1, C_2 ..., C_p such that $E(M) = C_1 \cup C_2 \cup ... \cup C_p$.

Definition 2.7. We call a matroid M bipartite, if every circuit of M has even cardinality.

Proposition 2.8. [8] *Let M be a binary matroid. Then M is Eulerian if and only if M^* is bipartite.*

To prove one of our main results, we need the following theorem.

Theorem 2.9. *Let M be a binary matroid and $T \subseteq E(M)$. Then $(M^*)_T = (M_T)^*$ if and only if T is a union of the disjoint cocircuits and also a union of the disjoint circuits of M.*

Proof. Suppose that T is a union of the disjoint cocircuits and a union of the disjoint circuits of M. Then by Lemma 2.2 and it's dual, $(M_T)^* = (M^*)_T$. So the result holds.

Conversely, suppose that $(M_T)^* = (M^*)_T$. We show that T is a union of the disjoint cocircuits and also a union of the disjoint circuits of M. Suppose this is not true. We consider the following cases:

Case (i): If T is neither a union of the disjoint cocircuits nor a union of the disjoint circuits of M. Then by Corollary 2.3, $r(M_T) = r(M) + 1$. So

$$
\begin{aligned}
r((M_T)^*) &= |E(M)| - (r(M)+1) \\
&= |E(M)| - r(M) - 1 \\
&= r(M^*) - 1 \qquad (1)
\end{aligned}
$$

On the other hand, by the dual of Corollary 2.3,

$$r((M^*)_T) = r(M^*) + 1 \qquad (2)$$

But, $|E(M)| - r(M) = r(M^*)$. Since $(M_T)^* = (M^*)_T$ so we have $r(M_T)^* = r(M^*)_T$; On the other hand by (1) and (2), we deduce that, $r(M^*) + 1 = r(M^*) - 1$, a contradiction.

Case (ii): If T is a union of the disjoint cocircuits but not a union of the disjoint circuits of M, then we have $(M_T)^* = M^*$, so

$$
\begin{aligned}
r((M_T)^*) &= r(M^*) \\
&= |E(M)| - r(M) \\
&= r(M^*) \qquad (3)
\end{aligned}
$$

On the other hand, by Lemma 2.2 along with the Corollary 2.2 applied to dual, we have

$$r((M^*)_T) = r(M^*) + 1 \qquad (4)$$

But $(M_T)^* = (M^*)_T$, therefore $r((M_T)^*) = r((M^*)_T)$. On the other hand by (3) and (4), we deduce that, $r(M^*) = r(M^*) + 1$, a contradiction.

Case (iii): If T is a union of the disjoint circuits but not the disjoint union of cocircuits of M, then by similar argument in case 2 on the dual of M, we get a contradiction, so the result holds. This completes the proof. □

Corollary 2.10. *Suppose M is a binary matroid on E and $T \subseteq E(M)$. Then $M|T$ is Eulerian and bipartite if and only if $(M^*)_T = (M_T)^*$.*

Proof. It is an immediate cosequence of the Theorem 2.9 and the Proposition 2.8. □

Corollary 2.11. *Let M be a n-connected binary matroid with at least $(2n-1)$ elements. Then for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* \neq (M^*)_T$.*

Proof. Suppose that there is a subset T with $|T| = n$ in which $(M_T)^* = (M^*)_T$. Then by Theorem 2.9, T is a union of the disjoint circuits and union of the disjoint cocircuits of M. So

$r(T) \leq n-k$ and $r^*(T) \leq n-j$ for some $j, k \geq 1$. On the other hand we have

$$\begin{aligned}\lambda_M(T) &= r(T) + r^*(T) - |T| \\ &= n-k+n-j-n \\ &\leq n-k-j.\end{aligned}$$

And since M has at least $(2n-1)$ elements, we conclude that $(T, E(M)-T)$ is a $(n-k-j+1)$-separation for M, a contradiction. □

Definition 2.12. Let F be an arbitrary field and $V=(v_1, v_2, ..., v_n)$ be a member of $V(n, F)$. The support of V is $\{i : v_i \neq 0\}$.

Proposition 2.13. [3] *Let M be a binary matroid on E. Then $E(M)$ can be partitioned into circuits if and only if there is a basis of the cocircuit space all of whose member have even support.*

Corollary 2.14. *Let M a binary matroid on E. Then M is eulerian if and only if there is a matrix A representing M and has even non-zero entries in each rows.*

Proposition 2.15. [3] *Let C be a circuit of a binary matroid M and e be an element of $E(M)-C$. Then, in M/e, either C is a circuit or C is a disjoint union of two circuits. In both case, M/e has no other circuits contained in C.*

The following corollary is an immediate consequence of Proposition 2.15.

Corollary 2.16. *2.16 Let M be binary matroid on E, $T \subsetneq E$ and $y \in E-T$. If T be a union of the disjoint cocircuits of M, then T is also a union of the disjoint cocircuits of $M|(T \cup \{y\})$.*

Theorem 2.17. *Let M be a binary matroid on E. Then for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* = (M^*)_T$ if and only if $n = |E(M)|$ and M is Eulerian and bipartite. Moreover, $n = 2k$ for some $k \in N$, that is, n is even.*

Proof. Since M is Eulerian, bipartite and $T = E(M)$, so by Theorem 2.9, $(M_T)^* = (M^*)_T$ and the result holds.

Conversely, suppose for every $T \subseteq E(M)$ with $|T| = n$, $(M_T)^* = (M^*)_T$. We prove that $T = E(M)$. Suppose $T \neq E(M)$ and $y \in E(M)-T$. Since $(M_T)^* = (M^*)_T$, so by Theorem 2.9 T is a union of the disjoint circuits and also a union of the disjoint cocircuits of M. Let A be a matrix that representing M, then by Corollary 2.14, in $M[A]|T$ the number of non-zero entries in each row is even. Suppose that $T \cup \{y\}$ is in the first columns of A.

set $T' = (T - \{x_1\}) \cup \{y\}$, where $x_1 \in T$. Now T' is an n-element subset of $E(M)$. But y must be parallel to x_1, otherwise in some row of $M[A]|T'$ in which y has non-zero entries, we have odd non-zero entries, a contradiction. Now there is an element $x_1 \neq x_i \in T$ which is not parallel to x_1 and y, otherwise $M|(T \cup \{y\}) \cong U_{1,|T|+1}$, this contradicts the Corollary 2.16.

Let $T'' = \{T \cup \{y\}\} - \{x_i\}$, where $x_1 \neq x_i \in T$. Now T'' is an n-element set of $E(M)$ in which $M|T''$ is not Eulerian, a contradiction. So T must be equal to $E(M)$. Now Theorem 2.9 and Proposition 2.9 show that M is Eulerian and bipartite.

Moreover since $E(M)$ is a union of the disjoint circuits, also a union of the disjoint cocircuits of M, and the fact that the intersection of circuits and cocircuits of a binary matroid has even element, we conclude that n is even. □

3. On Dual of the binary element splitting matroid

In this section, we show that $(M^*)'_T \neq (M'_T)^*$ for every $T \subseteq E(M)$ in the element splitting. As we pointed out in abstract, in this section we prove that for every $T \subseteq E(M)$, $(M^*)'_T \neq (M'_T)^*$.

Lemma 3.1. [6] *Let M be a binary matroid on E. Then for any $X \subseteq E(M)$, $r'(X \cup \{a\}) = r(X) + 1$ where r and r' denote the rank functions of M and M'_T, respectively.*

Corollary 3.2. *If M is a binary matroid on E and $T \subseteq E(M)$, then*

$$r'(M'_T) = r(M) + 1.$$

where r and r' denote the rank functions of M and M'_T, respectively.

Theorem 3.3. *Let M be a binary matroid on E and $T \subseteq E(M)$. Then for every $T \subseteq E(M)$, $(M^*)'_T \neq (M'_T)^*$.*

Proof. Suppose that there is a $T \subseteq E(M)$, such that $(M^*)'_T = (M'_T)^*$. Let r be the rank function. Then

$$r((M^*)'_T) = r((M'_T)^*). \tag{5}$$

But, by Corollary 3.2, $r(M'_T) = r(M) + 1$, so

$$\begin{aligned} r((M'_T)^*) &= |E(M) \cup \{a\}| - (r(M) + 1) \\ &= |E(M)| - r(M) \\ &= r(M^*). \end{aligned} \tag{6}$$

Again by Corollary 3.2,

$$r((M^*)'_T) = r(M^*) + 1. \tag{7}$$

Therefore by (5), (6) and (7) we have $r(M^*)+1=r(M^*)$, a contradiction. This completes the proof. $\square$

References

[1] S. B. Dhotre, *A note on the dual of the splitting matroid*, Lobachevskii J. Math., **33**, (2012), 229-231.

[2] H. Fleischner, *Eulerian Graphs and Related Topics*, North Holland, Amsterdam, (1990).

[3] J. G. Oxley, *Matroid Theory*, Oxford university press, New York, (2011).

[4] T. T. Raghunathan, M. M. Shikare and B. N. Waphare, *Splitting in a binary matroid*, Discrete math., **184**, (1998), 267-271.

[5] M. M. Shikare, Gh. Azadi, *Determination of the bases of a splitting matroid*, European J. combin., **24**, (2003), 45-52.

[6] M. M. Shikare, Gh. Azadi, B. N. Waphare, *Generalized splitting operation and its application*, J. Indian Math. Soc., **78**, (2011), 145-154.

[7] P. J. Slater, *A classsification of 4-connected graphs* , J. Combin. Theory, **17**, (1974), 281-298.

[8] D. J. A. Welsh, *Eulerian and bipartite matroids*, J. Comb. Theory, **6**, (1969), 375-377.

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