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ON ENDO-SEMIPRIME AND ENDO-COSEMIPRIME MODULES

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Abstract. In this paper, we study the notions of endo-semiprime and endo-cosemiprime modules and obtain some related results. For instance, we show that in a right self-injective ring R, all nonzero ideals of R are endo-semiprime as right (left) R-modules if and only if R is semiprime. Also, we prove that both being endo-semiprime and being endo-cosemiprime are Morita invariant properties.

1. Introduction

Throughout this paper, all rings have identity elements and all modules are right unitary. Unless otherwise stated, R denotes an arbitrary ring with identity element. If M is a right (resp., left) R-module, we use the notation M_R (resp., ${}_RM$). Let M be an R-module. If N is a submodule of M, we write $N \leq M$ and the annihilator of N (in R) is denoted by $\mathrm{ann}_R(N) = \{r \in R \mid Nr = 0\}$. Also, $N \leq M$ is called a fully invariant submodule of M if for every R-endomorphism $f : M \to M$, $f(N) \subseteq N$.

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An R-module M is said to be *quasi injective* if for any submodule N of M, any R-homomorphism from N to M can be extended to an endomorphism of M. A proper ideal P of a ring R is called a *prime* ideal of R if for any two ideals I and J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. Also, P is called a *semiprime* ideal of R if for any ideal I of R, $I^2 \subseteq P$ implies that $I \subseteq P$. The notion of prime ideals was extended from rings to modules by Dauns in [3]. In fact, a nonzero R-module M is called *prime* if $\text{ann}_R(M) = \text{ann}_R(N)$, for every nonzero submodule N of M. Also, a nonzero R-module M is called *semiprime* if $\text{ann}_R(N)$ is a semiprime ideal of R, for any nonzero submodule N of M. By a *prime* (resp., *semiprime*) *submodule* of a module M we mean a submodule N such that the module M/N is prime (resp., semiprime). It is easy to see that a proper submodule N of M is a prime submodule if for any submodule K of M and any ideal I of R, $KI \subseteq N$ implies that $K \subseteq N$ or $MI \subseteq N$. The dual of prime modules was introduced and studied by Ceken, Alkan and smith in [2]. In fact, a nonzero R-module M is called *coprime* if $\text{ann}_R(M) = \text{ann}_R(M/N)$, for any proper submodule N of M. Also, a nonzero R-module M is called *cosemiprime* if $\text{ann}_R(M/N)$ is a semiprime ideal of R, for any proper submodule N of M. It is easy to see that every prime (resp., coprime) R-module is semiprime (resp., cosemiprime). More details about these notions can be found in [1, 6, 8].

Let M be a right R-module and $S = \text{End}(M_R)$. In [5], the authors introduced and studied the notion of endo-prime modules. In fact, M is called *endo-prime* if for any nonzero fully invariant submodule N of M and any $f \in S$, $fN = 0$ implies that $f = 0$, i.e., $\text{ann}_S(N) = 0$, for any nonzero fully invariant submodule N of M. In this paper, we generalize this notion as follows: we say that M is *endo-semiprime* if $\text{ann}_S(N)$ is a semiprime ideal of S for any nonzero fully invariant submodule N of M. Also, we introduce and study the dual notion of endo-semiprime. A nonzero right R-module M is called *endo-cosemiprime* if $\text{ann}_S(M/N)$ is a semiprime ideal of S, for any proper fully invariant submodule N of M_R. Among other results, we prove that if M_R is epi-retractable (resp., co-mono-retractable) such that $S = \text{End}(M_R)$ is a semiprime ring, then M_R is endo-semiprime (resp., endo-cosemiprime) (Proposition 2.11). Also, it is shown that every semisimple module is both endo-semiprime and endo-cosemiprime (Corollary 2.14).

2. Endo-semiprime and endo-cosemiprime modules

Definition 2.1. Let M be a nonzero right R-module and $S = \text{End}(M_R)$.

(a) M is called endo-semiprime if for any nonzero fully invariant submodule N of M, $\text{ann}_S(N)$ is a semiprime ideal of S.

(b) M is called endo-cosemiprime if for any proper fully invariant submodule N of M, $\mathrm{ann}_S(M/N)$ is a semiprime ideal of S.

For example, if the integer number n is square-free, then $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}$-module is endo-semiprime. Because $n\mathbb{Z}$ is a semiprime ideal of $\mathbb{Z}$, see Corollary 2.7. Also, from the above definition, we can easily see that if M_R is endo-semiprime, then $S = \mathrm{End}(M_R)$ is a semiprime ring. We will show that $\mathbb{Z}_{p^\infty}$ is endo-cosemiprime while is not endo-semiprime, see Example 2.13. Also, it is easy to see that if M_R is endo-cosemiprime, then $S = \mathrm{End}(M_R)$ is a semiprime ring.

Proposition 2.2. *Let M be a right R-module and $S = \mathrm{End}(M_R)$. Then M_R is endo-semiprime (resp., endo-cosemiprime) if and only if ${}_SM$ is a semiprime (resp., cosemiprime) module.*

Proof. First suppose that M_R is endo-semiprime and $0 \neq N \leq {}_SM$. Then NR is a fully invariant submodule of M and by hypothesis, $\mathrm{ann}_S(NR)$ is a semiprime ideal of S. Clearly, $\mathrm{ann}_S(NR) = \mathrm{ann}_S(N)$ and this shows that ${}_SM$ is semiprime. Conversely, if ${}_SM$ is semiprime, then it is clear that M_R is endo-semiprime.
Now, let M_R be an endo-cosemiprime module and K be a proper submodule of ${}_SM$. We set $J = \mathrm{ann}_S(M/K)$. Then $JM \subseteq K \subsetneqq {}_SM$ and so JM is a proper fully invariant submodule of M_R. Since M_R is endo-cosemiprime, $\mathrm{ann}_S(M/JM)$ is semiprime. On the other hand, we have $J \subseteq \mathrm{ann}_S(M/JM) \subseteq \mathrm{ann}_S(M/K) = J$. Thus, $\mathrm{ann}_S(M/JM) = \mathrm{ann}_S(M/K)$ is semiprime and hence, ${}_SM$ is cosemiprime. Conversely, if K is a proper fully invariant submodule of M_R, then K is a proper submodule of ${}_SM$ and so by assumption $\mathrm{ann}_S(M/K)$ is semiprime. □

We need the following lemmas.

Lemma 2.3. *Let M be a right R-module and I be an ideal of R such that $MI = 0$. Then*
(1) $\mathrm{End}(M_R) = \mathrm{End}(M_{R/I})$;
(2) *M_R is endo-semiprime if and only if $M_{R/I}$ is endo-semiprime;*
(3) *For any submodule N of M, $\mathrm{ann}_R(N)$ is a semiprime ideal in R if and only if $\mathrm{ann}_{R/I}(N)$ is a semiprime ideal in R/I.*

Proof. (1) Since $mr = m(r + I)$, for any $r \in R$ and $m \in M$, we have $\mathrm{End}(M_R) = \mathrm{End}(M_{R/I})$.
(2) For any $N \subseteq M$, we have N is a fully invariant submodule of M_R if and only if N is a fully invariant submodule of $M_{R/I}$. Now, the result follows from part (1).
(3) We first assume that $\mathrm{ann}_R(N)$ is a semiprime ideal in R, where N is a submodule of M_R. Let $a \in R$ and $(a + I)R/I(a + I) \subseteq \mathrm{ann}_{R/I}(N)$. Then $N(a + I)R/I(a + I) = 0$ and so $N(a + I)(r + I)(a + I) = 0$, for any $r \in R$. Thus $N(ara + I) = Nara = 0$, for any $r \in R$. This implies that $NaRa = 0$ and since $\mathrm{ann}_R(N)$ is semiprime, $Na = 0$. Therefore, $N(a + I) = 0$

and so $a+I \in \mathrm{ann}_{R/I}(N)$. Thus $\mathrm{ann}_{R/I}(N)$ is a semiprime ideal in R/I. The argument for the converse is similar. □

Lemma 2.4. *Let M be a right R-module and I be an ideal of R such that $MI = 0$. Then*
(1) M_R is endo-cosemiprime if and only if $M_{R/I}$ is endo-cosemiprime;
(2) For any submodule N of M, $\mathrm{ann}_R(M/N)$ is a semiprime ideal in R if and only if $\mathrm{ann}_{R/I}(M/N)$ is a semiprime ideal in R/I.

Proof. By the equality $\mathrm{End}(M_R) = \mathrm{End}(M_{R/I})$, (1) is clear.
For see (2), we first assume that $\mathrm{ann}_R(M/N)$ is a semiprime ideal in R, where N is a submodule of M_R. Let $a \in R$ and $(a+I)R/I(a+I) \subseteq \mathrm{ann}_{R/I}(M/N)$. Then $M/N(a+I)R/I(a+I) = 0$ and so $M(a+I)(r+I)(a+I) \subseteq N$, for any $r \in R$. Thus $M(ara+I) = Mara \subseteq N$, for any $r \in R$. This implies that $MaRa \subseteq N$ and since $\mathrm{ann}_R(M/N)$ is semiprime, $Ma \subseteq N$. Therefore, $M(a+I) \subseteq N$ and so $a+I \in \mathrm{ann}_{R/I}(M/N)$. Thus $\mathrm{ann}_{R/I}(M/N)$ is a semiprime ideal in R/I. The argument for the converse is similar. □

For any two non-empty subsets A and B of a ring R, we denote the set $\{r \in R \mid rA \subseteq B\}$ by $(A : B)_l$.

Lemma 2.5. *Let I be a proper right ideal in a ring R. Then the cyclic right R-module R/I is endo-semiprime (resp., endo-cosemiprime) if and only if for any right ideal J that properly contains I and $(I : I)_l \subseteq (J : J)_l$ the following holds, for any $r \in R$:*

$$r(I : I)_l r \subseteq (J : I)_l \Rightarrow r \in (J : I)_l. \qquad (*)$$

$$\Big(resp., r(I : I)_l r \in (R : J)_l \Rightarrow r \in (R : J)_l\Big). \qquad (**)$$

Proof. It is easy to see that:

(1) $\mathrm{End}((R/I)_R) \cong (I : I)_l/I$.
(2) A submodule J/I of the right R-module R/I is fully invariant if and only if $(I : I)_l \subseteq (J : J)_l$.

Now, suppose that $(R/I)_R$ is endo-semiprime and $I \subsetneqq J$ is a right ideal of R such that $(I : I)_l \subseteq (J : J)_l$. By (2) in the above, J/I is a fully invariant submodule of $(R/I)_R$. Then its left annihilator in $\mathrm{End}((R/I)_R)$ is semiprime and so is in the ring $(I : I)_l/I$ by (1). This implies that for any $r+I \in (I : I)_l/I$, if $\Big((r+I)\frac{(I:I)_l}{I}(r+I)\Big) J/I = 0$, then $(r+I)J/I = 0$. In other words; if $(r(I : I)_l r)\, J \subseteq I$, then $rJ \subseteq I$. Conversely, let J/I be a fully invariant submodule in $(R/I)_R$. Then by (2), $(I : I)_l \subseteq (J : J)_l$ and by the relation $(*)$, the left annihilator of J/I is semiprime in $(I : I)_l/I$. So $(R/I)_R$ is endo-semiprime. For endo-cosemiprime, the proof is similar to the first part. □

Proposition 2.6. *Let I be a proper ideal in a ring R.*
(1) If $(R/I)_R$ is endo-semiprime, then for any ideal J in R that properly contains I, $(J : I)_l$ is a semiprime ideal in R.
(2) $(R/I)_R$ is endo-cosemiprime if and only if any ideal J that contains I, is semiprime.

Proof. (1) Let $I \subsetneqq J$ be an ideal of R and $(R/I)_R$ be endo-semiprime. Then by Lemma 2.3, $(R/I)_{R/I}$ is endo-semiprime and so $\text{ann}_S(J/I)$ is a semiprime ideal of S, where $S = \text{End}((R/I)_{R/I})$. Since $\text{End}((R/I)_{R/I}) \cong R/I$, we have $\text{ann}_{R/I}(J/I)$ is a semiprime ideal of R/I. Again by Lemma 2.3, $\text{ann}_R(J/I) = (J : I)_l$ is a semiprime ideal of R.
(2) It is an easy consequence of the Lemma 2.4 and this fact that $\text{End}(R_R) \cong R$, for any ring R. □

Corollary 2.7. *Let I be a proper ideal in a ring R. Then I is semiprime if and only if $(R/I)_R$ is endo-semiprime.*

Proof. If $(R/I)_R$ is endo-semiprime, then by setting $J = R$, in Proposition 2.6, we have $(R : I)_l = \{r \in R \mid rR \subseteq I\} = I$ is a semiprime ideal of R. Conversely, let I be semiprime and J be a right ideal in R such that properly contains I and $(I : I)_l \subseteq (J : J)_l$. Since $(I : I)_l = R$, we have $(J : J)_l = R$ and hence, J is a two-sided ideal of R. Now, suppose that $(r(I : I)_l r)J \subseteq I$. Then $rRrJ \subseteq I$ and since J is two-sided ideal, we have $RrRJRrRJ = RrRrRJ = RrRrJ \subseteq RI = I$. This implies that $RrRJ \subseteq I$. Thus, $rJ \subseteq I$ and by Lemma 2.5, $(R/I)_R$ is endo-semiprime. □

Now, the following result is immediate.

Corollary 2.8. *The following conditions are equivalent:*
(1) R is a semiprime ring;
(2) R_R is endo-semiprime;
(3) ${}_RR$ is endo-semiprime.

Corollary 2.9. (1) *R_R is endo-cosemiprime if and only if every proper ideal of R is semiprime.*
(2) If I is a right ideal in a ring R such that $(R/I)_R$ is an endo-semiprime R-module, then I behaves like a semiprime ideal, i.e., for any $a \in R$, $aRa \subseteq I$ concludes that $a \in I$.
(3) If R_R is endo-cosemiprime, then R is a semiprime ring.
(4) If R_R is endo-cosemiprime, then R_R is endo-semiprime.

Proof. (1) It follows from Proposition 2.6(2), by setting $I = 0$.
(2) Note that $(R : I)_l = I$ and $(I : I)_l \subseteq (R : R)_l = R$. If $aRa \subseteq I$, where $a \in R$, then by Lemma 2.5, $a \in (R : I)_l = I$.

(3) It follows from Corollary 2.6, because the zero ideal is a prime ideal in R.

(4) It follows from part (3) and Corollary 2.8. □

Remark 2.10. In Corollary 2.9, the converse of part (4) is not true in general. For example, $\mathbb{Z}_{\mathbb{Z}}$ is endo-semiprime because $\mathbb{Z}$ is a semiprime ring. But it is not endo-cosemiprime because $4\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$ that is not semiprime.

A right R-module M is called *retractable* if for any nonzero submodule N in M, $\mathrm{Hom}_R(M, N) \neq 0$ and M_R is called *epi-retractable* if for any nonzero submodule N in M, $Hom_R(M, N)$ contains a surjective element.

A right R-module M is called *co-retractable* if for any proper fully submodule K in M, $\mathrm{Hom}_R(M/K, M) \neq 0$. Also, an R-module M is called *co-mono-retractable* if for any proper submodule K in M, $\mathrm{Hom}_R(M/K, M)$ contains an injective element; equivalently there exists a nonzero homomorphism $h \in \mathrm{End}(M_R)$ such that $\mathrm{ker} h = K$. For more details see [4].

Proposition 2.11. *Let M_R be epi-retractable (resp., co-mono-retractable) such that $S = \mathrm{End}(M_R)$ is a semiprime ring. Then M_R is endo-semiprime (resp., endo-cosemiprime)*

Proof. First suppose that M_R is epi-retractable. Let N be a nonzero fully invariant submodule of M_R and $fSfN = 0$, where $f \in S = \mathrm{End}(M_R)$. By assumption, there exists $0 \neq g \in S$ such that $g(M) = N$. Thus, $fSfgM = 0$ and so $fgSfgM = 0$. Since S is semiprime, $fg = 0$ and hence, $fgM = fN = 0$.

For the second part, suppose that M_R is co-mono-retractable. Let K be a proper fully invariant submodule of M and $f \in S = \mathrm{End}(M_R)$ such that $fSf(M) \subseteq K$. By assumption, there exists a nonzero homomorphism $h \in S$ such that $\mathrm{ker} h = K$. Then $hfShf(M) \subseteq hfSf(M) \subseteq h(K) = 0$. Since S is semiprime, $hf(M) = 0$ and hence, $f(M) \subseteq \mathrm{ker} h = K$. □

Remark 2.12. The epi-retractable property is required in Proposition 2.11. For example, if p is a prime number, then $\mathbb{Z}_{p^\infty}$ is not epi-retractable $\mathbb{Z}$-module and its endomorphism ring is the integral domain of p-adic integers that is a semiprime ring, but $\mathbb{Z}_{p^\infty}$ is not an endo-semiprime $\mathbb{Z}$-module. Because if f is the homomorphism by multiplication p, then $f < \overline{1/p^2} > \neq 0$ whereas $f^2 < \overline{1/p^2} >= 0$.

Remark 2.10, together with the following example show that the concepts of endo-semiprime and endo-cosemiprime are independent conditions.

Example 2.13. $\mathbb{Z}_{p^\infty}$ is an endo-cosemiprime $\mathbb{Z}$-module. Because for any proper submodule K in $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty}/K \cong \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules. Thus, $\mathbb{Z}_{p^\infty}$ is co-mono-retractable and so by Proposition 2.11, $\mathbb{Z}_{p^\infty}$ is endo-cosemiprime. However, by Remark 2.12, $\mathbb{Z}_{p^\infty}$ is not endo-semiprime.

Corollary 2.14. *Every semisimple R-module is both endo-semiprime and endo-cosemiprime.*

Proof. Let M be a semisimple R-module. Then $\mathrm{End}(M_R) \cong \oplus_{\alpha\in A}\mathbb{RFM}_{\Gamma_\alpha}(D_\alpha)$, for some suitable division ring D_α and nonempty set Γ_α, where $\mathbb{RFM}_{\Gamma_\alpha}(D_\alpha)$ denotes a row finite Γ_α-matrix ring over ring D_α. We note that for any $\alpha \in A$, $\mathbb{RFM}_{\Gamma_\alpha}(D_\alpha)$ is a prime ring and so $\mathrm{End}(M_R)$ is semiprime. Now, M is endo-semiprime by Proposition 2.11. On the other hand, since any semisimple R-module is co-mono-retractable, by Proposition 2.11, M is endo-cosemiprime. □

The following example indicates that an endo-prime module is not necessarily an endo-semiprime module.

Example 2.15. Let $M = N_1 \oplus N_2$ be a semisimple module such that simple submodules N_1 and N_2 are not isomorphic. By Corollary 2.14, M is endo-semiprime but it is not endo-prime, because $\mathrm{End}(M_R) \cong \mathrm{End}(N_1) \oplus \mathrm{End}(N_2)$ is not prime.

Proposition 2.16. *Let M_R be an endo-semiprime R-module. If either R is a commutative ring or M is retractable, then M is semiprime.*

Proof. First assume that R is commutative, N is a nonzero submodule of M and $a \in R$ such that $a^2 \in \mathrm{ann}_R(N)$. We define R-homomorphism f as follows:

$$f : M \to M$$

$$f(x) = xa.$$

Then $f(SN) = SNa$ is a fully invariant submodule of M, where $S = \mathrm{End}(M_R)$. Thus, for any $h \in S$;

$$fhf(SN) = fh(SNa) = fh(SN)a \subseteq f(SN)a = (SNa)a = SNa^2 = 0,$$

and so $fSf(SN) = 0$. Since M_R is endo-semiprime and SN is a fully invariant submodule of M, $\mathrm{ann}_R(SN)$ is semiprime and hence, $Na = 0$, as desired.

Now, assume that M is retractable and N is a nonzero submodule of M such that $NI^2 = 0$ and $NI \neq 0$, for some ideal I of R. Then $SNI \neq 0$, where $S = \mathrm{End}(M_R)$. Since M is retractable, there exists a nonzero homomorphism $f \in S$ such that $f(M) \subseteq SNI$. Therefore, for any $h \in S$;

$$hf(M) \subseteq h(SNI) = h(SN)I \subseteq SNI.$$

Hence;

$$fhf(M) \subseteq f(SNI) = f(SN)I \subseteq f(M)I \subseteq (SNI)I = 0.$$

Consequently, we have $fSf = 0$ and since M is endo-semiprime, $f = 0$, a contradiction. Thus, $\mathrm{ann}_R(N)$ is semiprime. □

In [5], it is shown that if M is an endo-prime R-module, then the fully invariant submodules of M can not be summand. This fact is not true for endo-semiprime modules, because the $\mathbb{Z}$-modules $\mathbb{Z}_6$ is endo-semiprime and $3\mathbb{Z}_6$ is a fully invariant submodule in $\mathbb{Z}_6$ with $\mathbb{Z}_6 = 2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6$.

The direct sum of two endo-semiprime modules may be not endo-semiprime. To see this, consider the following example.

Example 2.17. Let p be a prime number. It is easy to see that $\mathbb{Z}$ and $\mathbb{Z}_p$ are endo-semiprime, as $\mathbb{Z}$-module. However, $\mathbb{Z} \oplus \mathbb{Z}_p$ is not endo-semiprime, because the ring

$$\mathrm{End}((\mathbb{Z} \oplus \mathbb{Z}_p)_{\mathbb{Z}}) \cong \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p \end{bmatrix}$$

is not semiprime.

In the following result we show that in some endo-semiprime modules, every fully invariant submodule is endo-semiprime.

Proposition 2.18. *Let M_R be an endo-semiprime R-module and N be a fully invariant submodule of M. If either N is a direct summand of M or M_R is quasi-injective, then N is an endo-semiprime R-module.*

Proof. If N is a direct summand of M, then it is easy to check that N is an endo-semiprime R-module. Now, assume that M_R is quasi-injective and K is a fully invariant submodule of N. Then K is also a fully invariant submodule of M. We set $S = \mathrm{End}(N_R)$ and $\overline{S} = \mathrm{End}(M_R)$. Suppose that $fSf(K) = 0$, for some $f \in S$. Then since M is quasi-injective, there exists $\overline{f} \in \overline{S}$ such that $\overline{f}|_N = f$. We show that $\overline{f}\ \overline{S}\ \overline{f}(K) = 0$. For each $\overline{h} \in \overline{S}$, $h = \overline{h}|_N \in S$ and since K is a fully invariant submodule of M we have:

$$\overline{f}\ \overline{h}\ \overline{f}(K) = \overline{f}\ \overline{h}f(K) = \overline{f}hf(K) = fhf(K) = 0.$$

Therefore, $\overline{f} = 0$, because M is endo-semiprime. So $f(K) = \overline{f}|_N(K) = 0$. □

Theorem 2.19. *Let R be a ring. Consider the following statements:*

(1) *R is semiprime.*

(2) *There exists a faithful retractable right (left) endo-semiprime R-module.*

(3) *All nonzero two-sided ideals of R are endo-semiprime as right (left) R-modules.*

Then (1) $\Leftrightarrow$ (2) *and* (3) $\Rightarrow$ (1). *Moreover; if R_R is injective, then* (1) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Let I be a nonzero right ideal of R and $0 \neq x \in I$. Then the map $f : R \to I$ defined by $f(r) = xr$ is a nonzero R-homomorphism. Thus, R_R is retractable. Since R is semiprime, by Corollary 2.8, R_R is endo-semiprime.

(2) $\Rightarrow$ (1). Let M be a faithful retractable endo-semiprime right R-module. By Proposition 2.16, for any nonzero submodule N in M, $\mathrm{ann}_R(N)$ is a semiprime ideal of R. Thus, $\mathrm{ann}_R(M) = 0$ is also semiprime. Consequently, R is a semiprime ring.
(3) $\Rightarrow$ (1) is trivial by Corollary 2.8.
(1) $\Rightarrow$ (3). Since R is semiprime, by Corollary 2.8, R_R is endo-semiprime. Now by Proposition 2.18, (3) is obtained because R_R is injective. □

Let M be a right R-module. A nonzero submodule N of M is called *essential* in M, denoted by $N \leq_e M$, if $N \cap K \neq 0$, for any nonzero submodule K of M. Also, the *singular submodule* of M is the submodule $Z(M) = \{m \in M \mid \ \mathrm{ann}_R(m) \leq_e R_R\}$. M is called *singular* (resp., *nonsingular*) if $Z(M) = M$ (resp., $Z(M) = 0$).

Remark 2.20. Let N be a nonzero fully invariant submodule of M. If M_R is nonsingular and $N \leq_e M$, then one can easily see that the restriction map $\varphi : \mathrm{End}(M_R) \to \mathrm{End}(N_R)$ is an injective homomorphism of rings, see [5, Lemma 1.8].

Proposition 2.21. *Let M be a quasi-injective nonsingular R-module and N be an essential fully invariant submodule of M. Then M_R is endo-semiprime if and only if N_R is endo-semiprime.*

Proof. The necessity is covered by Proposition 2.18. For sufficiency, suppose that N_R is endo-semiprime and K is a nonzero fully invariant submodule of M such that $fSf(K) = 0$, where $S = \mathrm{End}(M_R)$ and $f \in S$. By assumption, $N \leq_e M$ and so $N \cap K \neq 0$. Since both N and K are fully invariant, $N \cap K$ is also fully invariant. Now, as M is quasi-injective and $fSf(N \cap K) = 0$, we have $f|_N S' f|_N (N \cap K) = 0$, where $S' = \mathrm{End}(N_R)$. Thus, $f|_N(N \cap K) = 0$ and so $f|_{N\cap K}(N \cap K) = 0$. Since N is a fully invariant essential submodule of M, by Remark 2.20, $\varphi : \mathrm{End}(M) \to \mathrm{End}(N \cap K)$ is injective. Therefore, $f|_{N\cap K}(N \cap K) = 0$ implies that $f = 0$ and so $f(K) = 0$. □

In the following example, we show that the concepts of semiprime and endo-semiprime are independent conditions.

Example 2.22. (a) Let p be a prime number. By Example 2.17, the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_p$ is not endo-semiprime. However we show that M is a semiprime $\mathbb{Z}$-module. Let $0 \neq K \leq M$ and $I = n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ such that $KI^2 = 0$. If $n = 0$, then $KI = 0$. Thus, suppose that $n \neq 0$ and $(x, y) \in K$. Then $(x, y)n^2\mathbb{Z} = 0$ implies that $xn^2 = 0$ and $yn^2 = 0$; so $x = 0$ and p divides y or p divides n. In any case, we conclude that $(x, y)n\mathbb{Z} = 0$. Thus, $Kn\mathbb{Z} = 0$, as desired.

(b) Let K be a field and $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$, $J = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $J^2 = 0$, R is not semiprime. Now set $M = eR$. Then $\mathrm{End}(M_R) \cong eRe \cong K$ as rings, and hence, M is a semiprime left K-module. Thus, by Proposition 2.2, M_R is endo-semiprime. On the other hand, it is easy to see that $\mathrm{ann}_R(M) = 0$ and since R is not a semiprime ring, we have M_R is not semiprime.

Theorem 2.23. *Both being endo-semiprime and being endo-cosemiprime are Morita invariant properties.*

Proof. Suppose that A and B are Morita equivalent rings with inverse category equivalences $\alpha : Mod_A \to Mod_B$ and $\beta : Mod_B \to Mod_A$. First let M be an endo-semiprime object in Mod_A and N be a nonzero fully invariant submodule of $\alpha(M)$ with inclusion map i to $\alpha(M)$. Then $\beta(i)\beta(N)$ is a nonzero submodule of $\beta\alpha(M)$. Now, assume that $hTh(N) = 0$, for some $h \in T = \mathrm{End}(\alpha(M)_B)$. So for any f and f' in $\mathrm{End}(M_R)$, $h\alpha(f)h\alpha(f')i(N) = 0$. Then $\beta(h)\beta\alpha(f)\beta(h)\beta\alpha(f')\beta(i)(\beta(N)) = 0$. Thus, $\beta(h)U\beta(h)U\beta(i)(\beta(N)) = 0$, where $U = \mathrm{End}(\beta\alpha(M)_A)$. Since $\beta\alpha(M)$ is endo-semiprime and $U\beta(i)(\beta(N))$ is a nonzero fully invariant submodule of $\beta\alpha(M)$, $\beta(h)U\beta(i)(\beta(N)) = 0$. Then $\beta(h)\beta(i)(\beta(N)) = 0$ and so $h(N) = 0$.

Now, let M be an endo-cosemiprime object in Mod_A and N be a proper fully invariant submodule of $\alpha(M)$ with inclusion map i to $\alpha(M)$. Then $\beta(i)(\beta(N))$ is a proper submodule of $\beta\alpha(M)$. We set $J = \mathrm{ann}_U\left(\beta\alpha(M)/\beta(i)\beta(N)\right)$ where $U = \mathrm{End}((\beta\alpha(M))_A)$. Since $J\beta\alpha(M) \subseteq \beta(i)(\beta(N)) \subsetneqq \beta\alpha(M)$, then $J\beta\alpha(M)$ is a proper fully invariant submodule of $\beta\alpha(M)$. We show that $\mathrm{ann}_T(\alpha(M)/N)$ is semiprime, where $T = \mathrm{End}(\alpha(M)_B)$. Let $hTh\alpha(M) \subseteq N$, for some $h \in T$. Then for any $f \in \mathrm{End}(M_A)$, $h\alpha(f)h\alpha(M) \subseteq N = iN$. So $\beta(h)U\beta(h)\beta\alpha(M) \leq \beta(i)\beta(N)$. Therefore, $\beta(h)U\beta(h) \leq J$. So $\beta(h)U\beta(h)\beta\alpha(M) \leq J\beta\alpha(M)$. Since $\beta\alpha(M)$ is endo-cosemiprime, then $\beta(h)\beta\alpha(M) \leq J\beta\alpha(M) \leq \beta(i)\beta(N)$ and so $h\alpha(M) \leq N$. □

Now, we focus more on properties of endo-cosemiprime modules.

Proposition 2.24. *If R is a commutative ring and M is an endo-cosemiprime R-module, then $R/\mathrm{ann}_R(M)$ is a semiprime ring.*

Proof. We show that $\mathrm{ann}_R(M)$ is a semiprime ideal of R. Let $a \in R$ such that $a^2 \in \mathrm{ann}_R(M)$. Then $f : M \to M$ defined by $f(x) = xa$ is an R-homoorphism. Now, we have $fSf(M) = fS(Ma) \subseteq f(M)a = Ma^2 = 0$, where $S = \mathrm{End}(M_R)$. Since M is endo-cosemiprime, S is semiprime. Thus, $f(M) = 0$ and so $Ma = 0$. □

Proposition 2.25. *Let M_R be a co-mono-retractable module. If either M is nonsingular or every submodule of M is a projective R-module, then M is endo-cosemiprime.*

Proof. First assume that M is nonsingular and N is a submodule of M. If N is an essential submodule of M_R, then $(M/N)_R$ is singular. Since M is co-mono-retractable, there exists a monomorphism $f : M/N \to M$. Then $M/N \cong f(M/N) \subseteq M$ and so $Z(M) \cap f(M/N) = Z(f(M/N))$. Since $Z(M) = 0$ and $Z(M/N) = M/N$, we have $f(M/N) = 0$ and hence, $M/N = 0$. Thus, $M = N$ and this implies that M is semisimple; so it is endo-cosemiprime.

Now, suppose that every submodule of M is a projective R-module and N is a submodule of M. By assumption there exists a nonzero homomorphism $f \in \mathrm{End}(M_R)$ such that $\ker f = N$ and $M/N \cong \mathrm{Im} f$ is a projective submodule of M. Therefore, $0 \to N \to M \to M/N \to 0$ is a split short exact sequence, and so $M = N \oplus K$, for some submodule K of M. Thus, M is semisimple and by Corollary 2.14, it is endo-cosemiprime. □

Proposition 2.26. *Let M_R be endo-cosemiprime and $S = \mathrm{End}(M_R)$, then S_S is co-mono-retractable if and only if S is semisimple.*

Proof. Let S_S be co-mono-retractable. Since M_R is endo-cosemiprime, S is semiprime. So by [4, Corollary 1.7(7)], S is a semisimple ring. The converse is straightforward. □

Proposition 2.27. *Let R be a ring in which every two ideals are comparable. Then the followings are equivalent:*
(1) $\mathrm{ann}_R(M)$ *is semiprime;*
(2) $\mathrm{ann}_R(K) = \mathrm{ann}_R(M)$ *or* $\mathrm{ann}_R(M/K) = \mathrm{ann}_R(M)$*, for any nontrivial submodule K of M;*
(3) $\mathrm{ann}_R(K) = \mathrm{ann}_R(M)$ *or* $\mathrm{ann}_R(M/K) = \mathrm{ann}_R(M)$*, for any nontrivial fully invariant submodule K of M.*

Proof. (1) $\Rightarrow$ (2). Let K be a nontrivial submodule of M. By assumption, $\mathrm{ann}_R(K) \subseteq \mathrm{ann}_R(M/K)$ or $\mathrm{ann}_R(M/K) \subseteq \mathrm{ann}_R(K)$. If $\mathrm{ann}_R(K) \subseteq \mathrm{ann}_R(M/K)$, then $(\mathrm{ann}_R(K))^2 \subseteq \mathrm{ann}_R(M)$. For any $x \in \mathrm{ann}_R(K)$, we have $(xR)^2 \subseteq (\mathrm{ann}_R(K))^2 \subseteq \mathrm{ann}_R(M)$. Since by (1), $\mathrm{ann}_R(M)$ is semiprime, $xR \subseteq \mathrm{ann}_R(M)$ and so $x \in \mathrm{ann}_R(M)$. Thus, $\mathrm{ann}_R(K) = \mathrm{ann}_R(M)$. The other case is similar.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1) Let I be an ideal of R such that $MI^2 = 0$. If $MI = M$, or $MI = 0$, then $MI^2 = MI = 0$. Thus, we assume that MI is a nontrivial submodule of M. It is clear that MI is fully invariant. By (3), $\mathrm{ann}_R(MI) = \mathrm{ann}_R(M)$ or $\mathrm{ann}_R(M/MI) = \mathrm{ann}_R(M)$. If $\mathrm{ann}_R(MI) = \mathrm{ann}_R(M)$, then $I \subseteq \mathrm{ann}_R(MI) = \mathrm{ann}_R(M)$ and so $MI = 0$. If $\mathrm{ann}_R(M/MI) =$

$\mathrm{ann}_R(M)$, then $I \subseteq \mathrm{ann}_R(M/MI) = \mathrm{ann}_R(M)$. Thus, in any case, $MI^2 = MI = 0$, as desired.
□

Corollary 2.28. *Let R be a ring in which every two ideals are comparable and M be a faithful R-module. Then the following statements are equivalent:*
(1) *R is a semiprime ring;*
(2) $\mathrm{ann}_R(K) = 0$ *or* $\mathrm{ann}_R(M/K) = 0$*, for any nontrivial submodule K of M;*
(3) $\mathrm{ann}_R(K) = 0$ *or* $\mathrm{ann}_R(M/K) = 0$*, for any nontrivial fully invariant submodule K of M.*

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