ON ENDO-SEMIPRIME AND ENDO-COSEMIPRIME MODULES

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Abstract. In this paper, we study the notions of endo-semiprime and endo-cosemiprime modules and obtain some related results. For instance, we show that in a right self-injective ring $R$, all nonzero ideals of $R$ are endo-semiprime as right (left) $R$-modules if and only if $R$ is semiprime. Also, we prove that both being endo-semiprime and being endo-cosemiprime are Morita invariant properties.

1. Introduction

Throughout this paper, all rings have identity elements and all modules are right unitary. Unless otherwise stated, $R$ denotes an arbitrary ring with identity element. If $M$ is a right (resp., left) $R$-module, we use the notation $M_R$ (resp., $R_M$). Let $M$ be an $R$-module. If $N$ is a submodule of $M$, we write $N \leq M$ and the annihilator of $N$ (in $R$) is denoted by $\text{ann}_R(N) = \{ r \in R \mid Nr = 0 \}$. Also, $N \leq M$ is called a fully invariant submodule of $M$ if for every $R$-endomorphism $f : M \to M$, $f(N) \subseteq N$.

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An $R$-module $M$ is said to be \textit{quasi injective} if for any submodule $N$ of $M$, any $R$-homomorphism from $N$ to $M$ can be extended to an endomorphism of $M$. A proper ideal $P$ of a ring $R$ is called a \textit{prime} ideal of $R$ if for any two ideals $I$ and $J$ of $R$, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. Also, $P$ is called a \textit{semiprime} ideal of $R$ if for any ideal $I$ of $R$, $I^2 \subseteq P$ implies that $I \subseteq P$. The notion of prime ideals was extended from rings to modules by Dauns in [3]. In fact, a nonzero $R$-module $M$ is called \textit{prime} if $\text{ann}_R(M) = \text{ann}_R(N)$, for every nonzero submodule $N$ of $M$. Also, a nonzero $R$-module $M$ is called \textit{semiprime} if $\text{ann}_R(N)$ is a semiprime ideal of $R$, for any nonzero submodule $N$ of $M$. By a \textit{prime} (resp., \textit{semiprime}) submodule of a module $M$ we mean a submodule $N$ such that the module $M/N$ is prime (resp., semiprime). It is easy to see that a proper submodule $N$ of $M$ is a prime submodule if for any submodule $K$ of $M$ and any ideal $I$ of $R$, $KI \subseteq N$ implies that $K \subseteq N$ or $MI \subseteq N$. The dual of prime modules was introduced and studied by Ceken, Alkan and smith in [2]. In fact, a nonzero $R$-module $M$ is called \textit{coprime} if $\text{ann}_R(M) = \text{ann}_R(M/N)$, for any proper submodule $N$ of $M$. Also, a nonzero $R$-module $M$ is called \textit{cosemiprime} if $\text{ann}_R(M/N)$ is a semiprime ideal of $R$, for any proper submodule $N$ of $M$. It is easy to see that every prime (resp., coprime) $R$-module is semiprime (resp., cosemiprime). More details about these notions can be found in [1, 6, 8].

Let $M$ be a right $R$-module and $S = \text{End}(M_R)$. In [3], the authors introduced and studied the notion of endo-prime modules. In fact, $M$ is called \textit{endo-prime} if for any nonzero fully invariant submodule $N$ of $M$ and any $f \in S$, $fN = 0$ implies that $f = 0$, i.e., $\text{ann}_S(N) = 0$, for any nonzero fully invariant submodule $N$ of $M$. In this paper, we generalize this notion as follows: we say that $M$ is \textit{endo-semiprime} if $\text{ann}_S(N)$ is a semiprime ideal of $S$ for any nonzero fully invariant submodule $N$ of $M$. Also, we introduce and study the dual notion of endo-semiprime. A nonzero right $R$-module $M$ is called \textit{endo-cosemiprime} if $\text{ann}_S(M/N)$ is a semiprime ideal of $S$, for any proper fully invariant submodule $N$ of $M_R$. Among other results, we prove that if $M_R$ is epi-retractable (resp., co-mono-retractable) such that $S = \text{End}(M_R)$ is a semiprime ring, then $M_R$ is endo-semiprime (resp., endo-cosemiprime) (Proposition 2.11). Also, it is shown that every semisimple module is both endo-semiprime and endo-cosemiprime (Corollary 2.14).

2. \textbf{Endo-semiprime and endo-cosemiprime modules}

\textbf{Definition 2.1.} Let $M$ be a nonzero right $R$-module and $S = \text{End}(M_R)$.

(a) $M$ is called endo-semiprime if for any nonzero fully invariant submodule $N$ of $M$, $\text{ann}_S(N)$ is a semiprime ideal of $S$. 
(b) $M$ is called endo-cosemiprime if for any proper fully invariant submodule $N$ of $M$, $\text{ann}_S(M/N)$ is a semiprime ideal of $S$.

For example, if the integer number $n$ is square-free, then $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}$-module is endo-semiprime. Because $n\mathbb{Z}$ is a semiprime ideal of $\mathbb{Z}$, see Corollary 2.7. Also, from the above definition, we can easily see that if $M_R$ is endo-semiprime, then $S = \text{End}(M_R)$ is a semiprime ring. We will show that $\mathbb{Z}_{p^\infty}$ is endo-cosemiprime while is not endo-semiprime, see Example 2.10. Also, it is easy to see that if $M_R$ is endo-cosemiprime, then $S = \text{End}(M_R)$ is a semiprime ring.

**Proposition 2.2.** Let $M$ be a right $R$-module and $S = \text{End}(M_R)$. Then $M_R$ is endo-semiprime (resp., endo-cosemiprime) if and only if $SM$ is a semiprime (resp., cosemiprime) module.

**Proof.** First suppose that $M_R$ is endo-semiprime and $0 \neq N \leq SM$. Then $NR$ is a fully invariant submodule of $M$ and by hypothesis, $\text{ann}_S(NR)$ is a semiprime ideal of $S$. Clearly, $\text{ann}_S(NR) = \text{ann}_S(N)$ and this shows that $SM$ is semiprime. Conversely, if $SM$ is semiprime, then it is clear that $M_R$ is endo-semiprime.

Now, let $M_R$ be an endo-cosemiprime module and $K$ be a proper submodule of $SM$. We set $J = \text{ann}_S(M/K)$. Then $JM \subseteq K \subseteq SM$ and so $JM$ is a proper fully invariant submodule of $M_R$. Since $M_R$ is endo-cosemiprime, $\text{ann}_S(M/JM)$ is semiprime. On the other hand, we have $J \subseteq \text{ann}_S(M/JM) \subseteq \text{ann}_S(M/K) = J$. Thus, $\text{ann}_S(M/JM) = \text{ann}_S(M/K)$ is semiprime and hence, $SM$ is cosemiprime. Conversely, if $K$ is a proper fully invariant submodule of $M_R$, then $K$ is a proper submodule of $SM$ and so by assumption $\text{ann}_S(M/K)$ is semiprime.

We need the following lemmas.

**Lemma 2.3.** Let $M$ be a right $R$-module and $I$ be an ideal of $R$ such that $MI = 0$. Then

(1) $\text{End}(M_R) = \text{End}(M_{R/I})$;
(2) $M_R$ is endo-semiprime if and only if $M_{R/I}$ is endo-semiprime;
(3) For any submodule $N$ of $M$, $\text{ann}_R(N)$ is a semiprime ideal in $R$ if and only if $\text{ann}_{R/I}(N)$ is a semiprime ideal in $R/I$.

**Proof.** (1) Since $mr = m(r + I)$, for any $r \in R$ and $m \in M$, we have $\text{End}(M_R) = \text{End}(M_{R/I})$.
(2) For any $N \subseteq M$, we have $N$ is a fully invariant submodule of $M_R$ if and only if $N$ is a fully invariant submodule of $M_{R/I}$. Now, the result follows from part (1).
(3) We first assume that $\text{ann}_R(N)$ is a semiprime ideal in $R$, where $N$ is a submodule of $M_R$. Let $a \in R$ and $(a + I)R/I(a + I) \subseteq \text{ann}_{R/I}(N)$. Then $N(a + I)R/I(a + I) = 0$ and so $N(a + I)(r + I)(a + I) = 0$, for any $r \in R$. Thus $N(ara + I) = Nara = 0$, for any $r \in R$. This implies that $N\text{Ra} = 0$ and since $\text{ann}_R(N)$ is semiprime, $Na = 0$. Therefore, $N(a + I) = 0$. 


and so $a + I \in \text{ann}_{R/I}(N)$. Thus $\text{ann}_{R/I}(N)$ is a semiprime ideal in $R/I$. The argument for the converse is similar. \(\square\)

**Lemma 2.4.** Let $M$ be a right $R$-module and $I$ be an ideal of $R$ such that $MI = 0$. Then

1. $M$ is endo-cosemiprime if and only if $M_{R/I}$ is endo-cosemiprime;
2. For any submodule $N$ of $M$, $\text{ann}_{R}(M/N)$ is a semiprime ideal in $R$ if and only if $\text{ann}_{R/I}(M/N)$ is a semiprime ideal in $R/I$.

**Proof.** By the equality $\text{End}(M_{R}) = \text{End}(M_{R/I})$, (1) is clear.

For see (2), we first assume that $\text{ann}_{R}(M/N)$ is a semiprime ideal in $R$, where $N$ is a submodule of $M_{R}$. Let $a \in R$ and $(a + I)R/I(a + I) \subseteq \text{ann}_{R/I}(M/N)$. Then $M/N(a + I)R/I(a + I) = 0$ and so $M(a + I)(r + I)(a + I) \subseteq N$, for any $r \in R$. Thus $M(ara + I) = Marco \subseteq N$, for any $r \in R$. This implies that $MaRa \subseteq N$ and since $\text{ann}_{R}(M/N)$ is semiprime, $Ma \subseteq N$. Therefore, $M(a + I) \subseteq N$ and so $a + I \in \text{ann}_{R/I}(M/N)$. Thus $\text{ann}_{R/I}(M/N)$ is a semiprime ideal in $R/I$. The argument for the converse is similar. \(\square\)

For any two non-empty subsets $A$ and $B$ of a ring $R$, we denote the set $\{r \in R \mid rA \subseteq B\}$ by $(A : B)_{l}$.

**Lemma 2.5.** Let $I$ be a proper right ideal in a ring $R$. Then the cyclic right $R$-module $R/I$ is endo-semiprime (resp., endo-cosemiprime) if and only if for any right ideal $J$ that properly contains $I$ and $(I : I)_{l} \subseteq (J : J)_{l}$ the following holds, for any $r \in R$:

\[
\begin{align*}
    r(I : I)_{l}r \subseteq (J : I)_{l} & \Rightarrow r \in (J : I)_{l}. \quad (*) \\
    (\text{resp., } r(I : I)_{l}r \in (R : J)_{l} & \Rightarrow r \in (R : J)_{l}). \quad (**) 
\end{align*}
\]

**Proof.** It is easy to see that:

1. $\text{End}((R/I)_{R}) \cong (I : I)_{l}/I$.
2. A submodule $J/I$ of the right $R$-module $R/I$ is fully invariant if and only if $(I : I)_{l} \subseteq (J : J)_{l}$.

Now, suppose that $(R/I)_{R}$ is endo-semiprime and $I \nsubseteq J$ is a right ideal of $R$ such that $(I : I)_{l} \subseteq (J : J)_{l}$. By (2) in the above, $J/I$ is a fully invariant submodule of $(R/I)_{R}$. Then its left annihilator in $\text{End}((R/I)_{R})$ is semiprime and so is in the ring $(I : I)_{l}/I$ by (1). This implies that for any $r + I \in (I : I)_{l}/I$, if $\left((r + I)\frac{(r + I)}{I}\right) J/I = 0$, then $(r + I)J/I = 0$. In other words; if $(r(I : I)_{l}r) J \subseteq I$, then $rJ \subseteq I$. Conversely, let $J/I$ be a fully invariant submodule in $(R/I)_{R}$. Then by (2), $(I : I)_{l} \subseteq (J : J)_{l}$ and by the relation $(*)$, the left annihilator of $J/I$ is semiprime in $(I : I)_{l}/I$. So $(R/I)_{R}$ is endo-semiprime. For endo-cosemiprime, the proof is similar to the first part. \(\square\)
Proposition 2.6. Let $I$ be a proper ideal in a ring $R$.

(1) If $(R/I)_R$ is endo-semiprime, then for any ideal $J$ in $R$ that properly contains $I$, $(J : I)_I$ is a semiprime ideal in $R$.

(2) $(R/I)_R$ is endo-cosemiprime if and only if any ideal $J$ that contains $I$, is semiprime.

Proof. (1) Let $I \subsetneq J$ be an ideal of $R$ and $(R/I)_R$ be endo-semiprime. Then by Lemma 2.4, $(R/I)_{R/I}$ is endo-semiprime and so $\text{ann}_S(J/I)$ is a semiprime ideal of $S$, where $S = \text{End}((R/I)_{R/I})$. Since $\text{End}((R/I)_{R/I}) \cong R/I$, we have $\text{ann}_{R/I}(J/I)$ is a semiprime ideal of $R/I$. Again by Lemma 2.3, $\text{ann}_{R/I}(J/I) = (J : I)_I$ is a semiprime ideal of $R$.

(2) It is an easy consequence of the Lemma 2.4 and this fact that $\text{End}(R_R) \cong R$, for any ring $R$. □

Corollary 2.7. Let $I$ be a proper ideal in a ring $R$. Then $I$ is semiprime if and only if $(R/I)_R$ is endo-semiprime.

Proof. If $(R/I)_R$ is endo-semiprime, then by setting $J = R$, in Proposition 2.6, we have $(R : I)_I = \{ r \in R \mid rR \subseteq I \} = I$ is a semiprime ideal of $R$. Conversely, let $I$ be semiprime and $J$ be a right ideal in $R$ such that properly contains $I$ and $(I : J)_I \subseteq (J : J)_I$. Since $(I : I)_I = R$, we have $(J : J)_I = R$ and hence, $J$ is a two-sided ideal of $R$. Now, suppose that $(r(I : I)_I)r \subseteq I$. Then $rRJ \subseteq I$ and since $J$ is two-sided ideal, we have $RrJrRJ = RrRrJ \subseteq RI = I$. This implies that $rRJ \subseteq I$. Thus, $rJ \subseteq I$ and by Lemma 2.5, $(R/I)_R$ is endo-semiprime. □

Now, the following result is immediate.

Corollary 2.8. The following conditions are equivalent:

(1) $R$ is a semiprime ring;
(2) $R_R$ is endo-semiprime;
(3) $R_R$ is endo-semiprime.

Corollary 2.9. (1) $R_R$ is endo-cosemiprime if and only if every proper ideal of $R$ is semiprime.
(2) If $I$ is a right ideal in a ring $R$ such that $(R/I)_R$ is an endo-semiprime $R$-module, then $I$ behaves like a semiprime ideal, i.e., for any $a \in R$, $aRa \subseteq I$ concludes that $a \in I$.
(3) If $R_R$ is endo-cosemiprime, then $R$ is a semiprime ring.
(4) If $R_R$ is endo-cosemiprime, then $R_R$ is endo-semiprime.

Proof. (1) It follows from Proposition 2.6(2), by setting $I = 0$.
(2) Note that $(R : I)_I = I$ and $(I : I)_I \subseteq (R : R)_I = R$. If $aRa \subseteq I$, where $a \in R$, then by Lemma 2.3, $a \in (R : I)_I = I$. 

(3) It follows from Corollary 2.6 because the zero ideal is a prime ideal in $R$.

(4) It follows from part (3) and Corollary 2.8.

Remark 2.10. In Corollary 2.8, the converse of part (4) is not true in general. For example, $\mathbb{Z}$ is endo-semiprime because $\mathbb{Z}$ is a semiprime ring. But it is not endo-cosemiprime because $4\mathbb{Z}$ is a proper ideal of $\mathbb{Z}$ that is not semiprime.

A right $R$-module $M$ is called retractable if for any nonzero submodule $N$ in $M$, $\text{Hom}_R(M, N) \neq 0$ and $M$ is called epi-retractable if for any nonzero submodule $N$ in $M$, $\text{Hom}_R(M, N)$ contains a surjective element.

A right $R$-module $M$ is called co-retractable if for any proper fully submodule $K$ in $M$, $\text{Hom}_R(M/K, M) \neq 0$. Also, an $R$-module $M$ is called co-mono-retractable if for any proper submodule $K$ in $M$, $\text{Hom}_R(M/K, M)$ contains an injective element; equivalently there exists a nonzero homomorphism $h \in \text{End}(M_R)$ such that $\ker h = K$. For more details see [3].

Proposition 2.11. Let $M_R$ be epi-retractable (resp., co-mono-retractable) such that $S = \text{End}(M_R)$ is a semiprime ring. Then $M_R$ is endo-semiprime (resp., endo-cosemiprime).

Proof. First suppose that $M_R$ is epi-retractable. Let $N$ be a nonzero fully invariant submodule of $M_R$ and $fSfN = 0$, where $f \in S = \text{End}(M_R)$. By assumption, there exists $0 \neq g \in S$ such that $g(M) = N$. Thus, $fSfgM = 0$ and so $fgSfgM = 0$. Since $S$ is semiprime, $fg = 0$ and hence, $fgM = fN = 0$.

For the second part, suppose that $M_R$ is co-mono-retractable. Let $K$ be a proper fully invariant submodule of $M$ and $f \in S = \text{End}(M_R)$ such that $fSf(M) \subseteq K$. By assumption, there exists a nonzero homomorphism $h \in S$ such that $\ker h = K$. Then $hfSfh(M) \subseteq hfSf(M) \subseteq h(K) = 0$. Since $S$ is semiprime, $hf(M) = 0$ and hence, $f(M) \subseteq \ker h = K$. □

Remark 2.12. The epi-retractable property is required in Proposition 2.11. For example, if $p$ is a prime number, then $\mathbb{Z}_{p^\infty}$ is not epi-retractable $\mathbb{Z}$-module and its endomorphism ring is the integral domain of $p$-adic integers that is a semiprime ring, but $\mathbb{Z}_{p^\infty}$ is not an endo-semiprime $\mathbb{Z}$-module. Because if $f$ is the homomorphism by multiplication $p$, then $f < 1/p^2 > \neq 0$ whereas $f^2 < 1/p^2 > = 0$.

Remark 2.10, together with the following example show that the concepts of endo-semiprime and endo-cosemiprime are independent conditions.

Example 2.13. $\mathbb{Z}_{p^\infty}$ is an endo-cosemiprime $\mathbb{Z}$-module. Because for any proper submodule $K$ in $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty}/K \cong \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules. Thus, $\mathbb{Z}_{p^\infty}$ is co-mono-retractable and so by Proposition 2.11, $\mathbb{Z}_{p^\infty}$ is endo-cosemiprime. However, by Remark 2.12, $\mathbb{Z}_{p^\infty}$ is not endo-semiprime.
Corollary 2.14. Every semisimple $R$-module is both endo-semiprime and endo-cosemiprime.

Proof. Let $M$ be a semisimple $R$-module. Then $\text{End}(M) \cong \bigoplus_{\alpha \in A}\text{RFM}_{\Gamma_{\alpha}}(D_{\alpha})$, for some suitable division ring $D_{\alpha}$ and nonempty set $\Gamma_{\alpha}$, where $\text{RFM}_{\Gamma_{\alpha}}(D_{\alpha})$ denotes a row finite $\Gamma_{\alpha}$-matrix ring over ring $D_{\alpha}$. We note that for any $\alpha \in A$, $\text{RFM}_{\Gamma_{\alpha}}(D_{\alpha})$ is a prime ring and so $\text{End}(M)$ is semiprime. Now, $M$ is endo-semiprime by Proposition 2.11. On the other hand, since any semisimple $R$-module is co-mono-retractable, by Proposition 2.11, $M$ is endo-cosemiprime.

The following example indicates that an endo-prime module is not necessarily an endo-semiprime module.

Example 2.15. Let $M = N_1 \oplus N_2$ be a semisimple module such that simple submodules $N_1$ and $N_2$ are not isomorphic. By Corollary 2.14, $M$ is endo-semiprime but it is not endo-prime, because $\text{End}(M) \cong \text{End}(N_1) \oplus \text{End}(N_2)$ is not prime.

Proposition 2.16. Let $M_R$ be an endo-semiprime $R$-module. If either $R$ is a commutative ring or $M$ is retractable, then $M$ is semiprime.

Proof. First assume that $R$ is commutative, $N$ is a nonzero submodule of $M$ and $a \in R$ such that $a^2 \in \text{ann}_R(N)$. We define $R$-homomorphism $f$ as follows:

$$f : M \to M$$

$$f(x) = xa.$$

Then $f(SN) = SNa$ is a fully invariant submodule of $M$, where $S = \text{End}(M_R)$. Thus, for any $h \in S$;

$$fhf(SN) = fh(SNa) = fh(SN)a \subseteq f(SN)a = (SNa)a = SNa^2 = 0,$$

and so $fSf(SN) = 0$. Since $M_R$ is endo-semiprime and $SN$ is a fully invariant submodule of $M$, $\text{ann}_R(SN)$ is semiprime and hence, $Na = 0$, as desired.

Now, assume that $M$ is retractable and $N$ is a nonzero submodule of $M$ such that $NI^2 = 0$ and $NI \neq 0$, for some ideal $I$ of $R$. Then $SNI \neq 0$, where $S = \text{End}(M_R)$. Since $M$ is retractable, there exists a nonzero homomorphism $f \in S$ such that $f(M) \subseteq SNI$. Therefore, for any $h \in S$;

$$hf(M) \subseteq h(SNI) = h(SN)I \subseteq SNI.$$ 

Hence;

$$fhf(M) \subseteq f(SNI) = f(SN)I \subseteq f(M)I \subseteq (SNI)I = 0.$$

Consequently, we have $fSf = 0$ and since $M$ is endo-semiprime, $f = 0$, a contradiction. Thus, $\text{ann}_R(N)$ is semiprime. □
In [3], it is shown that if $M$ is an endo-prime $R$-module, then the fully invariant submodules of $M$ can not be summand. This fact is not true for endo-semiprime modules, because the $\mathbb{Z}$-modules $\mathbb{Z}_6$ is endo-semiprime and $3\mathbb{Z}_6$ is a fully invariant submodule in $\mathbb{Z}_6$ with $\mathbb{Z}_6 = 2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6$.

The direct sum of two endo-semiprime modules may be not endo-semiprime. To see this, consider the following example.

**Example 2.17.** Let $p$ be a prime number. It is easy to see that $\mathbb{Z}$ and $\mathbb{Z}_p$ are endo-semiprime, as $\mathbb{Z}$-module. However, $\mathbb{Z} \oplus \mathbb{Z}_p$ is not endo-semiprime, because the ring

$$\text{End}((\mathbb{Z} \oplus \mathbb{Z}_p)\mathbb{Z}) \cong \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p \end{bmatrix}$$

is not semiprime.

In the following result we show that in some endo-semiprime modules, every fully invariant submodule is endo-semiprime.

**Proposition 2.18.** Let $M_R$ be an endo-semiprime $R$-module and $N$ be a fully invariant submodule of $M$. If either $N$ is a direct summand of $M$ or $M_R$ is quasi-injective, then $N$ is an endo-semiprime $R$-module.

**Proof.** If $N$ is a direct summand of $M$, then it is easy to check that $N$ is an endo-semiprime $R$-module. Now, assume that $M_R$ is quasi-injective and $K$ is a fully invariant submodule of $N$. Then $K$ is also a fully invariant submodule of $M$. We set $S = \text{End}(N_R)$ and $\overline{S} = \text{End}(M_R)$. Suppose that $fSf(K) = 0$, for some $f \in S$. Then since $M$ is quasi-injective, there exists $\overline{f} \in \overline{S}$ such that $\overline{f}|_N = f$. We show that $\overline{f} \overline{S} \overline{f}(K) = 0$. For each $\overline{h} \in \overline{S}$, $h = \overline{h}|_N \in S$ and since $K$ is a fully invariant submodule of $M$ we have:

$$\overline{f} \overline{h} \overline{f}(K) = \overline{f} \overline{h}f(K) = \overline{h}f(K) = hf(K) = 0.$$

Therefore, $\overline{f} = 0$, because $M$ is endo-semiprime. So $f(K) = \overline{f}|_N(K) = 0$. \(\square\)

**Theorem 2.19.** Let $R$ be a ring. Consider the following statements:

1. $R$ is semiprime.
2. There exists a faithful retractable right (left) endo-semiprime $R$-module.
3. All nonzero two-sided ideals of $R$ are endo-semiprime as right (left) $R$-modules.

Then (1) $\iff$ (2) and (3) $\Rightarrow$ (1). Moreover; if $R_R$ is injective, then (1) $\Rightarrow$ (3).

**Proof.** (1) $\Rightarrow$ (2). Let $I$ be a nonzero right ideal of $R$ and $0 \neq x \in I$. Then the map $f : R \to I$ defined by $f(r) = xr$ is a nonzero $R$-homomorphism. Thus, $R_R$ is retractable. Since $R$ is semiprime, by Corollary 2.8, $R_R$ is endo-semiprime.
(2) ⇒ (1). Let $M$ be a faithful retractable endo-semiprime right $R$-module. By Proposition 2.16, for any nonzero submodule $N$ in $M$, $\text{ann}_R(N)$ is a semiprime ideal of $R$. Thus, $\text{ann}_R(M) = 0$ is also semiprime. Consequently, $R$ is a semiprime ring.

(3) ⇒ (1) is trivial by Corollary 2.8.

(1) ⇒ (3). Since $R$ is semiprime, by Corollary 2.8, $R$ is endo-semiprime. Now by Proposition 2.18, (3) is obtained because $R$ is injective.

Let $M$ be a right $R$-module. A nonzero submodule $N$ of $M$ is called essential in $M$, denoted by $N \leq_e M$, if $N \cap K \neq 0$, for any nonzero submodule $K$ of $M$. Also, the singular submodule of $M$ is the submodule $Z(M) = \{ m \in M \mid \text{ann}_R(m) \leq_e R_R \}$. $M$ is called singular (resp., nonsingular) if $Z(M) = M$ (resp., $Z(M) = 0$).

**Remark 2.20.** Let $N$ be a nonzero fully invariant submodule of $M$. If $M_R$ is nonsingular and $N \leq_e M$, then one can easily see that the restriction map $\varphi : \text{End}(M_R) \to \text{End}(N_R)$ is an injective homomorphism of rings, see [5, Lemma 1.8].

**Proposition 2.21.** Let $M$ be a quasi-injective nonsingular $R$-module and $N$ be an essential fully invariant submodule of $M$. Then $M_R$ is endo-semiprime if and only if $N_R$ is endo-semiprime.

**Proof.** The necessity is covered by Proposition 2.18. For sufficiency, suppose that $N_R$ is endo-semiprime and $K$ is a nonzero fully invariant submodule of $M$ such that $fSf(K) = 0$, where $S = \text{End}(M_R)$ and $f \in S$. By assumption, $N \leq_e M$ and so $N \cap K \neq 0$. Since both $N$ and $K$ are fully invariant, $N \cap K$ is also fully invariant. Now, as $M$ is quasi-injective and $fSf(N \cap K) = 0$, we have $f|_{N \cap K}(N \cap K) = 0$, where $S' = \text{End}(N_R)$. Thus, $f|_{N \cap K}(N \cap K) = 0$ and so $f|_{N \cap K}(N \cap K) = 0$. Since $N$ is a fully invariant essential submodule of $M$, by Remark 2.20, $\varphi : \text{End}(M) \to \text{End}(N \cap K)$ is injective. Therefore, $f|_{N \cap K}(N \cap K) = 0$ implies that $f = 0$ and so $f(K) = 0$. \(\square\)

In the following example, we show that the concepts of semiprime and endo-semiprime are independent conditions.

**Example 2.22.** (a) Let $p$ be a prime number. By Example 2.17, the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_p$ is not endo-semiprime. However we show that $M$ is a semiprime $\mathbb{Z}$-module. Let $0 \neq K \leq M$ and $J = n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ such that $KI^2 = 0$. If $n = 0$, then $KI = 0$. Thus, suppose that $n \neq 0$ and $(x, y) \in K$. Then $(x, y)n^2\mathbb{Z} = 0$ implies that $xn^2 = 0$ and $yn^2 = 0$; so $x = 0$ and $p$ divides $y$ or $p$ divides $n$. In any case, we conclude that $(x, y)n\mathbb{Z} = 0$. Thus, $Kn\mathbb{Z} = 0$, as desired.
(b) Let $K$ be a field and $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$, $J = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $J^2 = 0$, $R$ is not semiprime. Now set $M = eR$. Then $\text{End}(M_R) \cong eRe \cong K$ as rings, and hence, $M$ is a semiprime left $K$-module. Thus, by Proposition 222, $M_R$ is endo-semiprime. On the other hand, it is easy to see that $\text{ann}_R(M) = 0$ and since $R$ is not a semiprime ring, we have $M_R$ is not semiprime.

**Theorem 2.23.** Both being endo-semiprime and being endo-cosemiprime are Morita invariant properties.

**Proof.** Suppose that $A$ and $B$ are Morita equivalent rings with inverse category equivalences $\alpha : \text{Mod}_A \to \text{Mod}_B$ and $\beta : \text{Mod}_B \to \text{Mod}_A$. First let $M$ be an endo-semiprime object in $\text{Mod}_A$ and $N$ be a nonzero fully invariant submodule of $\alpha(M)$ with inclusion map $i$ to $\alpha(M)$. Then $\beta(i)\beta(N)$ is a nonzero submodule of $\beta\alpha(M)$. Now, assume that $hTh(N) = 0$, for some $h \in T = \text{End}(\alpha(M)_B)$. So for any $f$ and $f'$ in $\text{End}(M_R)$, $h\alpha(f)h\alpha(f')i(N) = 0$. Then $\beta(h)\beta\alpha(f)\beta(h)\beta\alpha(f')\beta(i)(\beta(N)) = 0$. Thus, $\beta(h)U\beta(h)\beta(i)(\beta(N)) = 0$, where $U = \text{End}(\beta\alpha(M)_A)$. Since $\beta\alpha(M)$ is endo-semiprime and $U\beta(i)(\beta(N))$ is a nonzero fully invariant submodule of $\beta\alpha(M)$, $\beta(h)U\beta(i)(\beta(N)) = 0$. Then $\beta(h)\beta(i)(\beta(N)) = 0$ and so $h(N) = 0$.

Now, let $M$ be an endo-cosemiprime object in $\text{Mod}_A$ and $N$ be a proper fully invariant submodule of $\alpha(M)$ with inclusion map $i$ to $\alpha(M)$. Then $\beta(i)(\beta(N))$ is a proper submodule of $\beta\alpha(M)$. We set $J = \text{ann}_U(\beta\alpha(M)/\beta(i)\beta(N))$ where $U = \text{End}((\beta\alpha(M))_A)$. Since $J\beta\alpha(M) \subseteq \beta(i)(\beta(N)) \subsetneq \beta\alpha(M)$, then $J\beta\alpha(M)$ is a proper fully invariant submodule of $\beta\alpha(M)$. We show that $\text{ann}_T(\alpha(M)/N)$ is semiprime, where $T = \text{End}(\alpha(M)_B)$. Let $hTh\alpha(M) \subseteq N$, for some $h \in T$. Then for any $f \in \text{End}(M_A)$, $h\alpha(f)h\alpha(M) \subseteq N = iN$. So $\beta(h)U\beta(h)\beta\alpha(M) \subseteq \beta(i)\beta(N)$. Therefore, $\beta(h)U\beta(h) \subseteq J$. So $\beta(h)U\beta(h)\beta\alpha(M) \subseteq J\beta\alpha(M)$. Since $\beta\alpha(M)$ is endo-cosemiprime, then $\beta(h)\beta\alpha(M) \subseteq J\beta\alpha(M) \subseteq \beta(i)\beta(N)$ and so $h\alpha(M) \subseteq N$. 

Now, we focus more on properties of endo-cosemiprime modules.

**Proposition 2.24.** If $R$ is a commutative ring and $M$ is an endo-cosemiprime $R$-module, then $R/\text{ann}_R(M)$ is a semiprime ring.

**Proof.** We show that $\text{ann}_R(M)$ is a semiprime ideal of $R$. Let $a \in R$ such that $a^2 \in \text{ann}_R(M)$. Then $f : M \to M$ defined by $f(x) = xa$ is an $R$-homorphism. Now, we have $fSf(M) = fS(Ma) \subseteq f(M)a = Ma^2 = 0$, where $S = \text{End}(M_R)$. Since $M$ is endo-cosemiprime, $S$ is semiprime. Thus, $f(M) = 0$ and so $Ma = 0$. 

**Proposition 2.25.** Let $M_R$ be a co-mono-retractable module. If either $M$ is nonsingular or every submodule of $M$ is a projective $R$-module, then $M$ is endo-cosemiprime.

**Proof.** First assume that $M$ is nonsingular and $N$ is a submodule of $M$. If $N$ is an essential submodule of $M_R$, then $(M/N)_R$ is singular. Since $M$ is co-mono-retractable, there exists a monomorphism $f : M/N \to M$. Then $M/N \cong f(M/N) \subseteq M$ and so $Z(M) \cap f(M/N) = Z(f(M/N))$. Since $Z(M) = 0$ and $Z(M/N) = M/N$, we have $f(M/N) = 0$ and hence, $M/N = 0$. Thus, $M = N$ and this implies that $M$ is semisimple; so it is endo-cosemiprime.

Now, suppose that every submodule of $M$ is a projective $R$-module and $N$ is a submodule of $M$. By assumption there exists a nonzero homomorphism $f \in \text{End}(M_R)$ such that $\ker f = N$ and $M/N \cong \text{Im} f$ is a projective submodule of $M$. Therefore, $0 \to N \to M \to M/N \to 0$ is a split short exact sequence, and so $M = N \oplus K$, for some submodule $K$ of $M$. Thus, $M$ is semisimple and by Corollary 2.24, it is endo-cosemiprime. □

**Proposition 2.26.** Let $M_R$ be endo-cosemiprime and $S = \text{End}(M_R)$, then $S_S$ is co-mono-retractable if and only if $S$ is semisimple.

**Proof.** Let $S_S$ be co-mono-retractable. Since $M_R$ is endo-cosemiprime, $S$ is semiprime. So by [1, Corollary 1.7(7)], $S$ is a semisimple ring. The converse is straightforward. □

**Proposition 2.27.** Let $R$ be a ring in which every two ideals are comparable. Then the followings are equivalent:

1. $\text{ann}_R(M)$ is semiprime;
2. $\text{ann}_R(K) = \text{ann}_R(M)$ or $\text{ann}_R(M/K) = \text{ann}_R(M)$, for any nontrivial submodule $K$ of $M$;
3. $\text{ann}_R(K) = \text{ann}_R(M)$ or $\text{ann}_R(M/K) = \text{ann}_R(M)$, for any nontrivial fully invariant submodule $K$ of $M$.

**Proof.** (1) ⇒ (2). Let $K$ be a nontrivial submodule of $M$. By assumption, $\text{ann}_R(K) \subseteq \text{ann}_R(M/K)$ or $\text{ann}_R(M/K) \subseteq \text{ann}_R(K)$. If $\text{ann}_R(K) \subseteq \text{ann}_R(M/K)$, then $(\text{ann}_R(K))^2 \subseteq \text{ann}_R(M)$. For any $x \in \text{ann}_R(K)$, we have $(xR)^2 \subseteq (\text{ann}_R(K))^2 \subseteq \text{ann}_R(M)$. Since by (1), $\text{ann}_R(M)$ is semiprime, $xR \subseteq \text{ann}_R(M)$ and so $x \in \text{ann}_R(M)$. Thus, $\text{ann}_R(K) = \text{ann}_R(M)$.

The other case is similar.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Let $I$ be an ideal of $R$ such that $MI^2 = 0$. If $MI = M$, or $MI = 0$, then $MI^2 = MI = 0$. Thus, we assume that $MI$ is a nontrivial submodule of $M$. It is clear that $MI$ is fully invariant. By (3), $\text{ann}_R(MI) = \text{ann}_R(M)$ or $\text{ann}_R(M/MI) = \text{ann}_R(M)$. If $\text{ann}_R(MI) = \text{ann}_R(M)$, then $I \subseteq \text{ann}_R(MI) = \text{ann}_R(M)$ and so $MI = 0$. If $\text{ann}_R(M/MI) = \ldots$
ann\(_R(M)\), then \(I \subseteq \text{ann}_R(M/MI) = \text{ann}_R(M)\). Thus, in any case, \(MI^2 = MI = 0\), as desired.

\[\square\]

**Corollary 2.28.** Let \(R\) be a ring in which every two ideals are comparable and \(M\) be a faithful \(R\)-module. Then the following statements are equivalent:

1. \(R\) is a semiprime ring;
2. \(\text{ann}_R(K) = 0\) or \(\text{ann}_R(M/K) = 0\), for any nontrivial submodule \(K\) of \(M\);
3. \(\text{ann}_R(K) = 0\) or \(\text{ann}_R(M/K) = 0\), for any nontrivial fully invariant sub-module \(K\) of \(M\).

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