



A SHORT NOTE ON PRIME SUBMODULES

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ABSTRACT. Let R be a commutative ring with identity and M be a unital R -module. A proper submodule N of M with $N :_R M = \mathfrak{p}$ is said to be prime or \mathfrak{p} -prime (\mathfrak{p} a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ or $r \in \mathfrak{p}$. In this paper we study a new equivalent conditions for a minimal prime submodules of an R -module to be a finite set, whenever R is a Noetherian ring. Also we introduce the concept of arithmetic rank of a submodule of a Noetherian module and we give an upper bound for it.

1. INTRODUCTION

Throughout this paper, let R be a commutative ring (with identity) and M be a unital R -module. A proper submodule N of M with $N :_R M = \mathfrak{p}$ is said to be prime or \mathfrak{p} -prime (\mathfrak{p} a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ or $r \in \mathfrak{p}$. An other equivalent notion of prime submodules was first introduced and systematically studied in [6]. Prime submodules have been studied by several authors; see, for example, [4], [1], [7], [9], [10], [11], [12] and [14]. In section 2, we prove some new results about the finiteness of the set of minimal prime submodules of an R -module. Also we introduce the concept of arithmetic rank

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of a submodule of a Noetherian module and we give an upper bound for it. Throughout, for any ideal \mathfrak{b} of R , the radical of \mathfrak{b} , denoted by $\text{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$ and we denote $\{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$, where $\text{Spec}(R)$ denotes the set of all prime ideals of R . The symbol \subseteq denotes containment and \subset denotes proper containment for sets. If N is a submodule of M , we write $N \leq M$. We denote the annihilator of a factor module M/N of M by $(N :_R M)$. The set of all maximal ideals of R is denoted by $\text{Max}(R)$. For any unexplained notation and terminology we refer the reader to [5], [13] and [16].

2. Preliminaries

The results of this section which will be useful in the next section given in [2].

Proposition 2.1. *Let R be ring and M be a non-zero R -module and N be a submodule of M . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be distinct prime ideals of R . Let for each $1 \leq i \leq n$, N_i be a \mathfrak{p}_i -prime submodule of M . If $N \subseteq \cup_{i=1}^n N_i$, then $N \subseteq N_j$ for some $1 \leq j \leq n$.*

Proof. We do induction on n . The case $n = 2$ is easy. Now let $n \geq 3$ and the case $n - 1$ is settled. By definition for each $1 \leq i \leq n$ we have $\mathfrak{p}_i = (N_i :_R M)$. From the hypothesis $N \subseteq \cup_{i=1}^n N_i$ it follows that $N = \cup_{i=1}^n (N_i \cap N)$. Now let the contrary be true. Then $N \not\subseteq N_i$ and hence $(N_i \cap N) \neq N$, for any $1 \leq i \leq n$. Also, from the inductive hypothesis it follows that $N \neq \cup_{i \in (\{1, \dots, n\} \setminus \{k\})} (N_i \cap N)$ for each $1 \leq k \leq n$ and so $(N_k \cap N) \not\subseteq \cup_{i \in (\{1, \dots, n\} \setminus \{k\})} (N_i \cap N)$. Let \mathfrak{q} be a minimal element of the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ with respect to " \subseteq ". Then $\mathfrak{p}_i \not\subseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \setminus \{\mathfrak{q}\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Let $J_i := (N_i :_R N)$, for all $i = 1, \dots, n$. Then from the definition it follows that $\mathfrak{p}_i \subseteq J_i$, for all $i = 1, \dots, n$. On the other hand for each $x \in N$ and $r \in R$, if $rx \in (N_i \cap N)$ and $x \notin (N_i \cap N)$, then $rx \in N_i$ and $x \notin N_i$. Therefore it follows from the definition that $r \in \mathfrak{p}_i$. So $rM \subseteq N_i$, and consequently, $rN \subseteq (N_i \cap N)$. As $(N_i \cap N) \neq N$ it follows that there exists an element $y \in (N \setminus (N_i \cap N))$. Now for each $s \in J_i$ we have $sy \in (N_i \cap N) \subseteq N_i$ and $y \notin N_i$. So it follows from the definition that $s \in \mathfrak{p}_i$. Therefore, $(N_i :_R N) = J_i = \mathfrak{p}_i = (N_i :_R M)$. But it is easy to see that $(N_i :_R N) = ((N_i \cap N) :_R N)$. Thus for each $1 \leq i \leq n$, $N_i \cap N$ is a \mathfrak{p}_i -prime submodule of N . Therefore without loss of generality we may assume that $N = M = \cup_{i=1}^n N_i$ and $N_n \not\subseteq \cup_{i=1}^{n-1} N_i$. Next let $T := \cap_{i=1}^n N_i$. Then it is not to see that for each $1 \leq i \leq n$, N_i/T is a \mathfrak{p}_i -prime submodule of M/T and $M/T = \cup_{i=1}^n N_i/T$. Therefore, without loss of generality we may assume $M = \cup_{i=1}^n N_i$ and $\cap_{i=1}^n N_i = 0$ and $N_n \not\subseteq \cup_{i=1}^{n-1} N_i$. Then there is an exact sequence $0 \rightarrow M \rightarrow \oplus_{i=1}^n M/N_i$, which implies that $\cap_{i=1}^n \mathfrak{p}_i = \text{Ann}_R(\oplus_{i=1}^n M/N_i) \subseteq \text{Ann}_R(M)$. On the other hand for each $1 \leq i \leq n$ we have $\text{Ann}_R(M) \subseteq (N_i :_R M) = \mathfrak{p}_i$. So $\text{Ann}_R(M) \subseteq \cap_{i=1}^n \mathfrak{p}_i$. Hence $\text{Ann}_R(M) = \cap_{i=1}^n \mathfrak{p}_i$. Now if we have $\cap_{i=1}^{n-1} N_i = 0$, then there is an exact sequence $0 \rightarrow M \rightarrow \oplus_{i=1}^{n-1} M/N_i$, which implies that

$\cap_{i=1}^{n-1} \mathfrak{p}_i = \text{Ann}_R(\oplus_{i=1}^{n-1} M/N_i) \subseteq \text{Ann}_R(M) = \cap_{i=1}^n \mathfrak{p}_i \subseteq \mathfrak{p}_n$. So $\mathfrak{p}_t \subseteq \mathfrak{p}_n$, for some $1 \leq t \leq n-1$, which is a contradiction. So $\cap_{i=1}^{n-1} N_i \neq 0$. Then there is an element $0 \neq a \in \cap_{i=1}^{n-1} N_i$. As $\cap_{i=1}^n N_i = 0$, it follows that $a \notin N_n$. On the other hand since $N_n \not\subseteq \cup_{i=1}^{n-1} N_i$, it follows that there is an element $b \in N_n$ such that $b \notin \cup_{i=1}^{n-1} N_i$. Now as $a + b \in \cup_{i=1}^n N_i$, it follows that $a + b \in N_k$ for some $1 \leq k \leq n$, which is a contradiction. This completes the inductive step. \square

Remark: Proposition 2.1 does not hold in general. For example let $p \geq 2$ be a prime number and $2 \leq n \in \mathbb{N}$. Let $R = \mathbb{Z}_p = \{\bar{0}, \bar{1}, \dots, \overline{p-1}\}$ and $M = \oplus_{i=1}^n \mathbb{Z}_p$. Let

$$\mathfrak{A} = \{N : N = Rx, \text{ for some } 0 \neq x \in M.\}.$$

Then \mathfrak{A} is a finite set that has at most 2^{p^n} element and for each $N \in \mathfrak{A}$, N is a $\{\bar{0}\}$ -prime submodule of M such that $M \subseteq \cup_{N \in \mathfrak{A}} N$. But $M \not\subseteq N$ for any $N \in \mathfrak{A}$. \square

The following proposition is a generalization of [13, Ex. 16.8].

Proposition 2.2. *Let R be a ring, M a non-zero R -module, N a submodule of M and $x \in M$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be distinct prime ideals of R . Let for each $1 \leq i \leq n$, N_i be a \mathfrak{p}_i -prime submodule of M . If $N + Rx \not\subseteq \cup_{i=1}^n N_i$, then there exists $a \in N$ such that $a + x \notin \cup_{i=1}^n N_i$.*

Proof. We use induction on n . Let $n = 1$. If $x \in N_1$ then $N \not\subseteq N_1$. So there is $a \in N \setminus N_1$ and it is easy to see that $a + x \notin N_1$. But if $x \notin N_1$, then by choosing $a = 0 \in N$ the assertion holds. Now suppose $n \geq 2$ and the case $n - 1$ is settled. Let \mathfrak{q} be a minimal element of the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ with respect to " \subseteq ". Then $\mathfrak{p}_i \not\subseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \setminus \{\mathfrak{q}\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Then it is easy to see that $\cap_{i=1}^{n-1} \mathfrak{p}_i \not\subseteq \mathfrak{p}_n$. By inductive hypothesis there is an element $b \in N$ such that $b + x \notin \cup_{i=1}^{n-1} N_i$. So the assertion hold for $a = b$, whenever $b + x \notin N_n$. So we may assume $b + x \in N_n$. Then we claim that $N \not\subseteq N_n$. Because, if $N \subseteq N_n$ then $x \in N_n$ and so $N + Rx \subseteq N_n \subseteq \cup_{i=1}^n N_i$, which is a contradiction. Therefore, there exists an element $c \in N \setminus N_n$. As $\cap_{i=1}^{n-1} \mathfrak{p}_i \not\subseteq \mathfrak{p}_n$ it follows that there exists an element $r \in (\cap_{i=1}^{n-1} \mathfrak{p}_i) \setminus \mathfrak{p}_n$. Then it easily follows from the definition of the \mathfrak{p}_n -prime submodule that $rc \notin N_n$. Moreover, since $r \in \cap_{i=1}^{n-1} \mathfrak{p}_i$ it follows from the definition that $rc \in \cap_{i=1}^{n-1} N_i$. Now it is easy to see that $rc + b + x \notin \cup_{i=1}^n N_i$. Therefore, the assertion hold for $a := rc + b \in N$. This completes the induction step. \square

Definition 2.3. Let R be a Noetherian ring and M be a finitely generated R -module. For each \mathfrak{p} -prime submodule N of M we define \mathfrak{p} -height of N as:

$$\mathfrak{p}\text{-ht}(N) := \sup\{k \in \mathbb{N}_0 : \exists N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \text{Spec}_{\mathfrak{p}}^R(M), \forall i\},$$

where $\text{Spec}_{\mathfrak{p}}^R(M)$ denotes to the set of all \mathfrak{p} -prime submodules of M as an R -module.

Definition 2.4. Let R be a Noetherian ring and M be a finitely generated R -module. For each \mathfrak{p} -prime submodule N of M we define height of N as:

$$\text{ht}(N) := \sup\{k \in \mathbb{N}_0 : \exists N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \text{Spec}_R(M), \forall i\},$$

where $\text{Spec}_R(M)$ denotes to the set of all prime submodules of M as an R -module.

3. MINIMAL PRIME SUBMODULES

The following lemma is needed in the proof of the first main result of this section. Note that in the sequel for any submodule B of an R -module M , the set of all minimal prime submodules of M over B is denoted by $\text{Min}(B)$. Moreover, we denote $\text{Min}(0)$ by $\text{Min}(M)$. Also, $V(B)$ is defined as follows:

$$V(B) := \{N \in \text{Spec}_R(M) : N \supseteq B\}.$$

Lemma 3.1. Let R be a commutative ring and $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(R)$. Let M be an R -module and $N_1, N_2 \in \text{Min}(M)$ be respectively \mathfrak{p} -prime and \mathfrak{q} -prime submodules. Then $N_1 \neq N_2$ if and only if $\mathfrak{p} \neq \mathfrak{q}$.

Proof. If $\mathfrak{p} \neq \mathfrak{q}$ then obviously $N_1 \neq N_2$. Conversely, Let $N_1 \neq N_2$ but $\mathfrak{p} = \mathfrak{q}$. Since $N_1 = \bigcap_{L \in \text{Spec}_{\mathfrak{p}}^R(M)} L$ and $N_2 = \bigcap_{L \in \text{Spec}_{\mathfrak{q}}^R(M)} L$, it follows that $N_1 = N_2$ which is a contradiction. \square

Definition 3.2. Let M be an R -module and B be a submodule of M . Set

$$D(B) := \{N \in \text{Min}(B) : N \text{ is not a finitely generated } R\text{-module}\}.$$

Definition 3.3. Let R be a Noetherian ring and M be a finitely generated R -module. Then we define $\dim \text{Spec}_R(M)$ as:

$$\dim \text{Spec}_R(M) := \sup\{\text{ht}(N) : N \in \text{Spec}_R(M)\}.$$

The minimal prime submodules of an R -module M has been studied in [17], for example see [17, Theorem 2.1]. In the next theorem we present a new conditions that an R -module M has only a finite number of minimal prime submodules, whenever R is a Noetherian ring, which is a generalization of [3, Theorem 2.1].

Theorem 3.4. *Let R be a Noetherian ring, M be an R -module and B be a submodule of M . Then the following statements are equivalent:*

- (1) $\text{Min}(B)$ is finite.
- (2) For every $\mathfrak{P} \in \text{Min}(B)$ there exists a finitely generated submodule $K_{\mathfrak{P}}$ of \mathfrak{P} such that $|V(K_{\mathfrak{P}}) \cap \text{Min}(B)| < \infty$.
- (3) For every $\mathfrak{P} \in \text{Min}(B)$ there exists a finitely generated submodule $N_{\mathfrak{P}}$ of \mathfrak{P} such that $V(N_{\mathfrak{P}}) \cap \text{Min}(B) = \{\mathfrak{P}\}$.
- (4) For every $\mathfrak{P} \in \text{Min}(B)$, $\mathfrak{P} \not\subseteq \bigcup_{L \in \text{Min}(B) \setminus \{\mathfrak{P}\}} L$.
- (5) For every $\mathfrak{P} \in \text{Min}(B)$ there exists an element $x_{\mathfrak{P}} \in \mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \text{Min}(B) = \{\mathfrak{P}\}$.
- (6) For every $\mathfrak{P} \in D(B)$, $\mathfrak{P} \not\subseteq \bigcup_{L \in \text{Min}(B) \setminus \{\mathfrak{P}\}} L$.
- (7) For every $\mathfrak{P} \in D(B)$ there exists an element $x_{\mathfrak{P}} \in \mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \text{Min}(B) = \{\mathfrak{P}\}$.
- (8) For every $\mathfrak{P} \in D(B)$ there exists a finitely generated submodule $K_{\mathfrak{P}}$ of \mathfrak{P} such that $|V(K_{\mathfrak{P}}) \cap \text{Min}(B)| < \infty$.
- (9) For every $\mathfrak{P} \in D(B)$ there exists a finitely generated submodule $N_{\mathfrak{P}}$ of \mathfrak{P} such that $V(N_{\mathfrak{P}}) \cap \text{Min}(B) = \{\mathfrak{P}\}$.

Proof. Without loss of generality, we may assume that $B = 0$, $\text{Spec}_R(M) \neq \emptyset$ and consequently $\text{Min}(M) \neq \emptyset$. (1) \Rightarrow (2) Since $\text{Min}(M)$ is finite, by Lemma 3.1 and Proposition 2.1, for every $\mathfrak{P} \in \text{Min}(M)$, $\mathfrak{P} \not\subseteq \bigcup_{L \in \text{Min}(M) \setminus \{\mathfrak{P}\}} L$ and so there exists $x \in \mathfrak{P} \setminus \bigcup_{L \in \text{Min}(M) \setminus \{\mathfrak{P}\}} L$. Set $K_{\mathfrak{P}} = Rx$. Then $K_{\mathfrak{P}}$ is finitely generated and the set $V(K_{\mathfrak{P}}) \cap \text{Min}(M) = \{\mathfrak{P}\}$ is finite. (2) \Rightarrow (3) Let $\mathfrak{P} \in \text{Min}(M)$ and $V(K_{\mathfrak{P}}) \cap \text{Min}(M) = \{\mathfrak{P}, \mathfrak{P}_2, \dots, \mathfrak{P}_n\}$. Using Lemma 2.1 and Proposition 2.1 we can find an element $x \in \mathfrak{P} \setminus \bigcup_{i=2}^n \mathfrak{P}_i$. Let $N_{\mathfrak{P}} := K_{\mathfrak{P}} + Rx$. Then $N_{\mathfrak{P}}$ is finitely generated and $V(N_{\mathfrak{P}}) \cap \text{Min}(M) = \{\mathfrak{P}\}$. (3) \Rightarrow (1) Suppose the contrary be true. Then the set $\text{Min}(M)$ is infinite. Let

$$A := \{\mathfrak{p} \in \text{Spec}(R) : \text{Spec}_R^{\mathfrak{p}}(M) \cap \text{Min}(M) \neq \emptyset\}$$

$$E := \{N \leq M : N \text{ is finitely generated and } V(N) \cap \text{Min}(M) \text{ is a finite set.}\}$$

$$F := \{L \leq M : \forall N \in E, N \not\subseteq L\}$$

We show that there exists a maximal element K of F such that $(K :_R M)$ is a prime ideal. Since $\text{Min}(M)$ is infinite, so the zero submodule of M belong to the F and therefore by Zorn's Lemma F has a maximal element. Let L be a maximal element of F . If $(L :_R M)$ be a prime ideal, we are through. If not, then it is clear that $(L :_R M) \neq R$. Let $\mathfrak{q}_1 \in \text{Ass}_R(R/(L :_R M))$. By the definition there exists $r \in R \setminus (L :_R M)$ such that $\mathfrak{q}_1 = ((L :_R M) : r)$ and therefore $\mathfrak{q}_1 r M \subseteq L$. Since $r \notin (L :_R M)$, it follows that there exists an element $x \in M$ such that

$rx \notin L$. Now there exists $N \in E$ such that $N \subseteq L + Rrx$. In particular,

$$\mathfrak{q}_1 N \subseteq L + \mathfrak{q}_1 rx \subseteq L + \mathfrak{q}_1 rM \subseteq L.$$

Since $\mathfrak{q}_1 N$ is finitely generated, so $|V(\mathfrak{q}_1 N) \cap \text{Min}(M)| = \infty$. But in this case for all $\mathfrak{P} \in (V(\mathfrak{q}_1 N) \cap \text{Min}(M)) \setminus (V(N) \cap \text{Min}(M))$, we have $\mathfrak{q}_1 N \subseteq \mathfrak{P}$ and $N \not\subseteq \mathfrak{P}$. Now if \mathfrak{P} be a \mathfrak{p} -Prime submodule, then $\mathfrak{q}_1 \subseteq \mathfrak{p}$ and so $|V(\mathfrak{q}_1) \cap A| = \infty$. Hence $|V(\mathfrak{q}_1 M) \cap \text{Min}(M)| = \infty$. So for all $N \in E$, we have $N \not\subseteq \mathfrak{q}_1 M$ and therefore $\mathfrak{q}_1 M \in F$. Let

$$U := \{\mathfrak{q} \in V(\mathfrak{q}_1) : \mathfrak{q}M \in F\}.$$

Since R is Noetherian it follows that U has a maximal element, say \mathfrak{q}_2 . But $\mathfrak{q}_2 M \subseteq H$, for some maximal element H of F . We claim that $(H :_R M)$ is a prime ideal of R . If not, according to the above argument, there exists $\mathfrak{q}_3 \in \text{Ass}_R(R/(H :_R M))$ such that $\mathfrak{q}_3 M \in F$ and $\mathfrak{q}_2 \subseteq (H :_R M) \subseteq \mathfrak{q}_3$. By choosing \mathfrak{q}_2 , we must have $\mathfrak{q}_2 = \mathfrak{q}_3$, which is a contradiction. Therefore $(H :_R M) = \mathfrak{q}_2$ is a prime ideal. Now we show that H is a \mathfrak{q}_2 -prime submodule. Otherwise there exist $x \in M \setminus H$ and $r \in R \setminus \mathfrak{q}_2$, such that $rx \in H$. So $r \in Z_R(M/H) = \bigcup_{\mathfrak{q} \in \text{Ass}_R(M/H)} \mathfrak{q}$ and hence there exists $\mathfrak{q}' \in \text{Ass}_R(M/H)$ such that $r \in \mathfrak{q}'$. Consequently, $\mathfrak{q}_2 \subset \mathfrak{q}'$. On the other hand by the definition $\mathfrak{q}' = (H :_R y)$ for some $y \in M \setminus H$. Since $H \subset H + Ry$, it follows that there exists $N \in E$ such that $N \subseteq H + Ry$ and so $\mathfrak{q}' N \subseteq H$. According to the above argument, $|V(\mathfrak{q}' M) \cap \text{Min}(M)| = \infty$ which implies $\mathfrak{q}' M \in F$. Finally, we have $\mathfrak{q}_2 = (H :_R M) \subset \mathfrak{q}'$, which is a contradiction with the choosing \mathfrak{q}_2 . Therefore H is a \mathfrak{q}_2 -prime submodule of M . Whence, H contains a minimal prime submodule of M such as \mathfrak{P} . By assumption there exists a submodule $N_{\mathfrak{P}}$ of \mathfrak{P} such that $N_{\mathfrak{P}} \subseteq \mathfrak{P} \subseteq H$ and $N_{\mathfrak{P}} \in E$, which is a contradiction. Therefore, $\text{Min}(M)$ is a finite set.

Now the proof of (1) \Leftrightarrow (2) \Leftrightarrow (3) is complete.

(1) \Rightarrow (4) Follows from Lemma 3.1 and Proposition 2.1.

(4) \Rightarrow (1) \Leftrightarrow (5) Since (5) \Leftrightarrow (4) \Rightarrow (3) is clear so we have (1) \Leftrightarrow (4) \Leftrightarrow (5).

Now we have the following: (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5).

(4) \Rightarrow (6) Is clear.

(6) \Rightarrow (3) Since for every $\mathfrak{P} \in D(0)$, $\mathfrak{P} \not\subseteq \bigcup_{L \in \text{Min}(M) \setminus \{\mathfrak{P}\}} L$, it follows that there exists $x_{\mathfrak{P}} \in \mathfrak{P}$ such that $V(Rx_{\mathfrak{P}}) \cap \text{Min}(M) = \{\mathfrak{P}\}$. On the other hand for all $\mathfrak{P} \in (\text{Min}(M) \setminus D(0))$, we have $V(\mathfrak{P}) \cap \text{Min}(M) = \{\mathfrak{P}\}$, where \mathfrak{P} is finitely generated. So the assertion follows.

(6) \Leftrightarrow (7) and (1) \Rightarrow (8), (9) are clear.

(8), (9) \Rightarrow (3) Follow by a similar arguments as in (6) \Rightarrow (3). \square

The following results follow from Theorem 3.4.

Corollary 3.5. *Let R be a Noetherian ring, M an R -module and B be a proper submodule of M . Then $\text{Min}(B)$ is infinite if and only if there exists $\mathfrak{P} \in D(B)$ such that $\mathfrak{P} \subseteq \bigcup_{L \in (\text{Min}(B) \setminus \{\mathfrak{P}\})} L$.*

Proof. Follows immediately from Theorem 3.4. \square

Corollary 3.6. *Let R be a Noetherian ring, M an R -module and B be a proper submodule of M such that any minimal prime submodule over B is finitely generated. Then $\text{Min}(B)$ is finite.*

Proof. Follows immediately from Theorem 3.4. \square

Definition 3.7. Let R be a Noetherian ring, $M \neq 0$ be a finitely generated R -module and N be a proper submodule of M . Then *the radical of N* is defined as:

$$\text{Rad}(N) = \bigcap_{L \in \text{Min } N} L.$$

Before bringing the next definition, recall that for any ideal I of a Noetherian ring, the *arithmetic rank* of I , denoted by $\text{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I , i.e.,

$$\text{ara}(I) := \min\{n \in \mathbb{N}_0 : \exists x_1, \dots, x_n \in I \text{ with } \text{Rad}((x_1, \dots, x_n)) = \text{Rad}(I)\}.$$

Definition 3.8. Let R be a Noetherian ring, $M \neq 0$ be a finitely generated R -module and N be a proper submodule of M . We define the *arithmetic rank* of N , as:

$$\text{ara}(N) := \min\{n \in \mathbb{N}_0 : \exists x_1, \dots, x_n \in N \text{ with } \text{Rad}((x_1, \dots, x_n)) = \text{Rad}(N)\}.$$

The next theorem is a generalization of [15, Theorem 2.7].

Theorem 3.9. *Let R be a Noetherian ring, $M \neq 0$ a finitely generated R -module and N be a proper submodule of M . Then $\text{ara}(N) \leq \dim \text{Spec}_R(M) + 1$*

Proof. Let $d := \dim \text{Spec}_R(M)$. We may assume that d is finite. Now, suppose, to the contrary, that $\text{ara}(N) > d + 1$. Let $n := \text{ara}(N)$. Since $n > d + 1 \geq 1$ it follows from the definition that there exist elements x_1, \dots, x_n in N such that $\text{Rad}(N) = \text{Rad}((x_1, \dots, x_n))$. As $n > 0$ it follows that $\text{Min}(0) \setminus V(N) \neq \emptyset$. Therefore it follows from Lemma 3.1 and proposition 2.1 that

$N \not\subseteq \bigcup_{L \in \text{Min}(0) \setminus V(N)} L$. Therefore $(x_1, \dots, x_n) \not\subseteq \bigcup_{L \in \text{Min}(0) \setminus V(N)} L$, and so by Proposition 2.2 there is $a_1 \in (x_2, \dots, x_n)$ such that

$$x_1 + a_1 \notin \bigcup_{L \in \text{Min}(0) \setminus V(N)} L.$$

Let $y_1 := x_1 + a_1$. Then $y_1 \in N$ and $\text{Rad}(N) = \text{Rad}((y_1, x_2, \dots, x_n))$. We shall construct the sequence $y_1, \dots, y_{n-1} \in N$ such that $\text{Rad}(N) = \text{Rad}((y_1, \dots, y_{n-1}, x_n))$ and $y_j \notin \bigcup_{L \in \text{Min}((y_1, \dots, y_{j-1})) \setminus V(N)} L$, for each $1 \leq j \leq n - 1$, by an inductive process. To do this end, assume that $1 \leq k < n - 1$, and that we have already constructed elements y_1, \dots, y_k such that

$$\text{Rad}(N) = \text{Rad}((y_1, \dots, y_k, x_{k+1}, \dots, x_n)).$$

We show how to construct y_{k+1} . To do this, as $k < n - 1$ it follows that

$$\text{Min}((y_1, \dots, y_k)) \setminus V(N) \neq \emptyset.$$

Therefore it follows from Lemma 3.1 and proposition 2.1 that

$$N \not\subseteq \bigcup_{L \in \text{Min}((y_1, \dots, y_k)) \setminus V(N)} L.$$

Therefore $(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \not\subseteq \bigcup_{L \in \text{Min}((y_1, \dots, y_k)) \setminus V(N)} L$, and so by Proposition 2.2 there is $a_{k+1} \in (y_1, \dots, y_k, x_{k+2}, \dots, x_n)$ such that

$$x_{k+1} + a_{k+1} \notin \bigcup_{L \in \text{Min}((y_1, \dots, y_k)) \setminus V(N)} L.$$

Let $y_{k+1} := x_{k+1} + a_{k+1}$. Then $y_{k+1} \in N$ and $\text{Rad}(N) = \text{Rad}((y_1, \dots, y_k, y_{k+1}, x_{k+2}, \dots, x_n))$. This completes the inductive step in the construction. Now it is easy to see that $\text{Min}((y_1, \dots, y_{n-1})) \setminus V(N) \neq \emptyset$. Also, by using an induction argument we can deduce that for any $1 \leq j \leq n - 1$ and any $L \in \text{Min}((y_1, \dots, y_j)) \setminus V(N)$ we have $\text{ht}(L) \geq j$. Consequently, since there exists a prime submodule L of M in which $L \in \text{Min}((y_1, \dots, y_{n-1})) \setminus V(N)$ it follows that $n - 1 \leq \text{ht}(L) \leq \dim \text{Spec}_R(M) = d$, which implies that $n \leq d + 1$, as required. \square

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