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A SHORT NOTE ON PRIME SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be a unital R-module. A proper submodule N of M with $N :_R M = \mathfrak{p}$ is said to be prime or $\mathfrak{p}$-prime ($\mathfrak{p}$ a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ or $r \in \mathfrak{p}$. In this paper we study a new equivalent conditions for a minimal prime submodules of an R-module to be a finite set, whenever R is a Noetherian ring. Also we introduce the concept of arithmetic rank of a submodule of a Noetherian module and we give an upper bound for it.

1. Introduction

Throughout this paper, let R be a commutative ring (with identity) and M be a unital R-module. A proper submodule N of M with $N :_R M = \mathfrak{p}$ is said to be prime or $\mathfrak{p}$-prime ($\mathfrak{p}$ a prime ideal of R) if $rx \in N$ for $r \in R$ and $x \in M$ implies that either $x \in N$ or $r \in \mathfrak{p}$. An other equivalent notion of prime submodules was first introduced and systematically studied in [6]. Prime submodules have been studied by several authors; see, for example, [4], [1], [7], [9], [10], [11], [12] and [14]. In section 2, we prove some new results about the finiteness of the set of

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minimal prime submodules of an R-module. Also we introduce the concept of arithmetic rank of a submodule of a Noetherian module and we give an upper bound for it. Throughout, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$ and we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$, where $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R. The symbol $\subseteq$ denotes containment and $\subset$ denotes proper containment for sets. If N is a submodule of M, we write $N \leq M$. We denote the annihilator of a factor module M/N of M by $(N :_R M)$. The set of all maximal ideals of R is denoted by $\mathrm{Max}(R)$. For any unexplained notation and terminology we refer the reader to [5], [13] and [16].

2. **Preliminaries**

The results of this section which will be useful in the next section given in [2].

Proposition 2.1. *Let R be ring and M be a non-zero R-module and N be a submodule of M. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n$ be distinct prime ideals of R. Let for each $1 \leq i \leq n$, N_i be a $\mathfrak{p}_i$-prime submodule of M. If $N \subseteq \cup_{i=1}^{n} N_i$, then $N \subseteq N_j$ for some $1 \leq j \leq n$.*

Proof. We do induction on n. The case $n = 2$ is easy. Now let $n \geq 3$ and the case $n-1$ is settled. By definition for each $1 \leq i \leq n$ we have $\mathfrak{p}_i = (N_i :_R M)$. From the hypothesis $N \subseteq \cup_{i=1}^{n} N_i$ it follows that $N = \cup_{i=1}^{n}(N_i \cap N)$. Now let the contrary be true. Then $N \nsubseteq N_i$ and hence $(N_i \cap N) \neq N$, for any $1 \leq i \leq n$. Also, from the inductive hypothesis it follows that $N \neq \cup_{i\in(\{1,\ldots,n\}\setminus\{k\})}(N_i \cap N)$ for each $1 \leq k \leq n$ and so $(N_k \cap N) \nsubseteq \cup_{i\in(\{1,\ldots,n\}\setminus\{k\})}(N_i \cap N)$. Let $\mathfrak{q}$ be a minimal element of the set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ with respect to " $\subseteq$ ". Then $\mathfrak{p}_i \nsubseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}\setminus\{q\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Let $J_i := (N_i :_R N)$, for all $i = 1, ..., n$. Then from the definition it follows that $\mathfrak{p}_i \subseteq J_i$, for all $i = 1, ..., n$. On the other hand for each $x \in N$ and $r \in R$, if $rx \in (N_i \cap N)$ and $x \notin (N_i \cap N)$, then $rx \in N_i$ and $x \notin N_i$. Therefore it follows from the definition that $r \in \mathfrak{p}_i$. So $rM \subseteq N_i$, and consequently, $rN \subseteq (N_i \cap N)$. As $(N_i \cap N) \neq N$ it follows that there exists an element $y \in (N\setminus(N_i \cap N))$. Now for each $s \in J_i$ we have $sy \in (N_i \cap N) \subseteq N_i$ and $y \notin N_i$. So it follows from the definition that $s \in \mathfrak{p}_i$. Therefore, $(N_i :_R N) = J_i = \mathfrak{p}_i = (N_i :_R M)$. But it is easy to see that $(N_i :_R N) = ((N_i \cap N) :_R N)$. Thus for each $1 \leq i \leq n$, $N_i \cap N$ is a $\mathfrak{p}_i$-prime submodule of N. Therefore without loss of generality we may assume that $N = M = \cup_{i=1}^{n} N_i$ and $N_n \nsubseteq \cup_{i=1}^{n-1} N_i$. Next let $T := \cap_{i=1}^{n} N_i$. Then it is not to see that for each $1 \leq i \leq n$, N_i/T is a $\mathfrak{p}_i$-prime submodule of M/T and $M/T = \cup_{i=1}^{n} N_i/T$. Therefore, without loss of generality we may assume $M = \cup_{i=1}^{n} N_i$ and $\cap_{i=1}^{n} N_i = 0$ and $N_n \nsubseteq \cup_{i=1}^{n-1} N_i$. Then there is an exact sequence $0 \to M \to \oplus_{i=1}^{n} M/N_i$, which implies that $\cap_{i=1}^{n} \mathfrak{p}_i = \mathrm{Ann}_R(\oplus_{i=1}^{n} M/N_i) \subseteq \mathrm{Ann}_R(M)$. On the other hand for each $1 \leq i \leq n$ we have $\mathrm{Ann}_R(M) \subseteq (N_i :_R M) = \mathfrak{p}_i$. So $\mathrm{Ann}_R(M) \subseteq \cap_{i=1}^{n} \mathfrak{p}_i$. Hence $\mathrm{Ann}_R(M) = \cap_{i=1}^{n} \mathfrak{p}_i$. Now if we have $\cap_{i=1}^{n-1} N_i = 0$, then there is an exact sequence $0 \to M \to \oplus_{i=1}^{n-1} M/N_i$, which implies that $\cap_{i=1}^{n-1} \mathfrak{p}_i = \mathrm{Ann}_R(\oplus_{i=1}^{n-1} M/N_i) \subseteq \mathrm{Ann}_R(M) = \cap_{i=1}^{n} \mathfrak{p}_i \subseteq \mathfrak{p}_n$. So $\mathfrak{p}_t \subseteq \mathfrak{p}_n$, for some $1 \leq t \leq n-1$, which is a contradiction. So $\cap_{i=1}^{n-1} N_i \neq 0$. Then there is an element $0 \neq a \in \cap_{i=1}^{n-1} N_i$. As $\cap_{i=1}^{n} N_i = 0$, it follows that $a \notin N_n$. On the other hand since $N_n \nsubseteq \cup_{i=1}^{n-1} N_i$, it follows that there is an element $b \in N_n$ such that $b \notin \cup_{i=1}^{n-1} N_i$. Now as $a + b \in \cup_{i=1}^{n} N_i$, it follows that $a + b \in N_k$ for some $1 \leq k \leq n$, which is a contradiction. This completes the inductive step. $\square$

Remark: Proposition 2.1 does not hold in general. For example let $p \geq 2$ be a prime number and $2 \leq n \in \mathbb{N}$. Let $R = \mathbb{Z}_p = \{\overline{0}, \overline{1}, ..., \overline{p-1}\}$ and $M = \oplus_{i=1}^{n} \mathbb{Z}_p$. Let

$$\mathfrak{A} = \{N \ : \ N = Rx, \text{ for some } 0 \neq x \in M.\}.$$

Then $\mathfrak{A}$ is a finite set that has at most 2^{p^n} element and for each $N \in \mathfrak{A}$, N is a $\{\overline{0}\}$-prime submodule of M such that $M \subseteq \cup_{N \in \mathfrak{A}} N$. But $M \nsubseteq N$ for any $N \in \mathfrak{A}$. $\square$

The following proposition is a generalization of [13, Ex. 16.8].

Proposition 2.2. *Let R be a ring, M a non-zero R-module, N a submodule of M and $x \in M$. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n$ be distinct prime ideals of R. Let for each $1 \leq i \leq n$, N_i be a $\mathfrak{p}_i$-prime submodule of M. If $N + Rx \nsubseteq \cup_{i=1}^{n} N_i$, then there exists $a \in N$ such that $a + x \notin \cup_{i=1}^{n} N_i$.*

Proof. We use induction on n. Let $n = 1$. If $x \in N_1$ then $N \nsubseteq N_1$. So there is $a \in N \backslash N_1$ and it is easy to see that $a + x \notin N_1$. But if $x \notin N_1$, then by choosing $a = 0 \in N$ the assertion holds. Now suppose $n \geq 2$ and the case $n-1$ is settled. Let $\mathfrak{q}$ be a minimal element of the set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ with respect to " $\subseteq$ ". Then $\mathfrak{p}_i \nsubseteq \mathfrak{q}$ for each $\mathfrak{p}_i \in (\{\mathfrak{p}_1, ..., \mathfrak{p}_n\} \backslash \{q\})$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}_n$. Then it is easy to see that $\cap_{i=1}^{n-1} \mathfrak{p}_i \nsubseteq \mathfrak{p}_n$. By inductive hypothesis there is an element $b \in N$ such that $b + x \notin \cup_{i=1}^{n-1} N_i$. So the assertion hold for $a = b$, whenever $b + x \notin N_n$. So we may assume $b + x \in N_n$. Then we claim that $N \nsubseteq N_n$. Because, if $N \subseteq N_n$ then $x \in N_n$ and so $N + Rx \subseteq N_n \subseteq \cup_{i=1}^{n} N_i$, which is a contradiction. Therefore, there exists an element $c \in N \backslash N_n$. As $\cap_{i=1}^{n-1} \mathfrak{p}_i \nsubseteq \mathfrak{p}_n$ it follows that there exists an element $r \in (\cap_{i=1}^{n-1} \mathfrak{p}_i) \backslash \mathfrak{p}_n$. Then it easily follows from the definition of the $\mathfrak{p}_n$-prime submodule that $rc \notin N_n$. Moreover, since $r \in \cap_{i=1}^{n-1} \mathfrak{p}_i$ it follows from the definition that $rc \in \cap_{i=1}^{n-1} N_i$. Now it is easy to see that $rc + b + x \notin \cup_{i=1}^{n} N_i$. Therefore, the assertion hold for $a := rc + b \in N$. This completes the induction step. $\square$

Definition 2.3. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p}$-prime submodule N of M we define $\mathfrak{p}$-*height of* N as:

$$\mathfrak{p}\text{-ht}(N) := \sup\{k \in \mathbb{N}_0 \ : \ \exists\ N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \mathrm{Spec}_R^{\mathfrak{p}}(M), \ \forall\, i\},$$

where $\mathrm{Spec}_R^{\mathfrak{p}}(M)$ denotes to the set of all $\mathfrak{p}$-prime submodules of M as an R-module.

Definition 2.4. Let R be a Noetherian ring and M be a finitely generated R-module. For each $\mathfrak{p}$-prime submodule N of M we define *height of* N as:

$$\mathrm{ht}(N) := \sup\{k \in \mathbb{N}_0 \ : \ \exists\ N_0 \subset \cdots \subset N_k = N, \text{ with } N_i \in \mathrm{Spec}_R(M), \ \forall\, i\},$$

where $\mathrm{Spec}_R(M)$ denotes to the set of all prime submodules of M as an R-module.

3. Minimal prime submodules

The following lemma is needed in the proof of the first main result of this section. Note that in the sequel for any submodule B of an R-module M, the set of all minimal prime submodules of M over B is denoted by $\mathrm{Min}(B)$. Moreover, we denote $\mathrm{Min}(0)$ by $\mathrm{Min}(M)$. Also, $V(B)$ is defined as follows:

$$V(B) := \{N \in \mathrm{Spec}_R(M) \ : \ N \supseteq B\}.$$

Lemma 3.1. *Let R be a commutative ring and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$. Let M be an R-module and $N_1, N_2 \in \mathrm{Min}(M)$ be respectively $\mathfrak{p}$-prime and $\mathfrak{q}$-prime submodules. Then $N_1 \neq N_2$ if and only if $\mathfrak{p} \neq \mathfrak{q}$.*

Proof. If $\mathfrak{p} \neq \mathfrak{q}$ then obviously $N_1 \neq N_2$. Conversely, Let $N_1 \neq N_2$ but $\mathfrak{p} = \mathfrak{q}$. Since $N_1 = \bigcap_{L \in \mathrm{Spec}_R^{\mathfrak{p}}(M)} L$ and $N_2 = \bigcap_{L \in \mathrm{Spec}_R^{\mathfrak{q}}(M)} L$, it follows that $N_1 = N_2$ which is a contradiction.□

Definition 3.2. Let M be an R-module and B be a submodule of M. Set

$$D(B) := \{N \in \mathrm{Min}(B) \ : \ N \text{ is not } a \text{ finitely generated } R-\text{module}\}.$$

Definition 3.3. Let R be a Noetherian ring and M be a finitely generated R-module. Then we define $dimSpec_R(M)$ as:

$$\mathrm{dimSpec}_R(M) := \sup\{\mathrm{ht}(N) \ : \ N \in \mathrm{Spec}_R(M)\}.$$

The minimal prime submodules of an $R-$module M has been studied in [17], for example see [17, Theorem 2.1]. In the next theorem we present a new conditions that an $R-$module M has only a finite number of minimal prime submodules, whenever R is a Noetherian ring, which is a generalization of [3, Theorem 2.1].

Theorem 3.4. *Let R be a Noetherian ring, M be an R-module and B be a submodule of M. Then the following statements are equivalent:*
(1) $\mathrm{Min}(B)$ *is finite.*
(2) For every $\mathfrak{P} \in \mathrm{Min}(B)$ *there exists a finitely generated submodule* $K_{\mathfrak{P}}$ *of* $\mathfrak{P}$ *such that* $|V(K_{\mathfrak{P}}) \bigcap \mathrm{Min}(B)| < \infty$.
(3) For every $\mathfrak{P} \in \mathrm{Min}(B)$ *there exists a finitely generated submodule* $N_{\mathfrak{P}}$ *of* $\mathfrak{P}$ *such that* $V(N_{\mathfrak{P}}) \bigcap \mathrm{Min}(B) = \{\mathfrak{P}\}$.
(4) For every $\mathfrak{P} \in \mathrm{Min}(B)$, $\mathfrak{P} \nsubseteq \bigcup_{L \in \mathrm{Min}(B) \setminus \{\mathfrak{P}\}} L$.
(5) For every $\mathfrak{P} \in \mathrm{Min}(B)$ *there exists an element* $x_{\mathfrak{P}} \in \mathfrak{P}$ *such that* $V(Rx_{\mathfrak{P}}) \bigcap \mathrm{Min}(B) = \{\mathfrak{P}\}$.
(6) For every $\mathfrak{P} \in D(B)$, $\mathfrak{P} \nsubseteq \bigcup_{L \in \mathrm{Min}(B) \setminus \{\mathfrak{P}\}} L$.
(7) For every $\mathfrak{P} \in D(B)$ *there exists an element* $x_{\mathfrak{P}} \in \mathfrak{P}$ *such that* $V(Rx_{\mathfrak{P}}) \bigcap \mathrm{Min}(B) = \{\mathfrak{P}\}$.
(8) For every $\mathfrak{P} \in D(B)$ *there exists a finitely generated submodule* $K_{\mathfrak{P}}$ *of* $\mathfrak{P}$ *such that* $|V(K_{\mathfrak{P}}) \bigcap \mathrm{Min}(B)| < \infty$.
(9) For every $\mathfrak{P} \in D(B)$ *there exists a finitely generated submodule* $N_{\mathfrak{P}}$ *of* $\mathfrak{P}$ *such that* $V(N_{\mathfrak{P}}) \bigcap \mathrm{Min}(B) = \{\mathfrak{P}\}$.

Proof. Without loss of generality, we may assume that $B = 0$, $\mathrm{Spec}_R(M) \neq \emptyset$ and consequently $\mathrm{Min}(M) \neq \emptyset$. (1) $\Rightarrow$ (2) Since $\mathrm{Min}(M)$ is finite, by Lemma 3.1 and Proposition 2.1, for every $\mathfrak{P} \in \mathrm{Min}(M)$, $\mathfrak{P} \nsubseteq \bigcup_{L \in \mathrm{Min}(M) \setminus \{\mathfrak{P}\}} L$ and so there exists $x \in \mathfrak{P} \setminus \bigcup_{L \in \mathrm{Min}(M) \setminus \{\mathfrak{P}\}} L$. Set $K_{\mathfrak{P}} = Rx$. Then $K_{\mathfrak{P}}$ is finitely generated and the set $V(K_{\mathfrak{P}}) \bigcap \mathrm{Min}(M) = \{\mathfrak{P}\}$ is finite.
(2) $\Rightarrow$ (3) Let $\mathfrak{P} \in \mathrm{Min}(M)$ and $V(K_{\mathfrak{P}}) \bigcap \mathrm{Min}(M) = \{\mathfrak{P}, \mathfrak{P}_2, ..., \mathfrak{P}_n\}$. Using Lemma 2.1 and Proposition 2.1 we can find an element $x \in \mathfrak{P} \setminus \bigcup_{i=2}^{n} \mathfrak{P}_i$. Let $N_{\mathfrak{P}} := K_{\mathfrak{P}} + Rx$. Then $N_{\mathfrak{P}}$ is finitely generated and $V(N_{\mathfrak{P}}) \bigcap \mathrm{Min}(M) = \{\mathfrak{P}\}$.
(3) $\Rightarrow$ (1) Suppose the contrary be true. Then the set $\mathrm{Min}(M)$ is infinite. Let

$$A := \{\mathfrak{p} \in \mathrm{Spec}(R) \ : \ \mathrm{Spec}_R^{\mathfrak{p}}(M) \cap \mathrm{Min}(M) \neq \emptyset\}$$

$$E := \{N \leq M \ : \ N \text{ is finitely generated and } V(N) \cap \mathrm{Min}(M) \text{ is a finite set.}\}$$

$$F := \{L \leq M \ : \ \forall N \in E, \ N \nsubseteq L\}$$

We show that there exists a maximal element K of F such that $(K :_R M)$ is a prime ideal. Since $\mathrm{Min}(M)$ is infinite, so the zero submodule of M belong to the F and therefore by Zorn's Lemma F has a maximal element. Let L be a maximal element of F. If $(L :_R M)$ be a prime ideal, we are through. If not, then it is clear that $(L :_R M) \neq R$. Let $\mathfrak{q}_1 \in \mathrm{Ass}_R(R/(L :_R M))$. By the definition there exists $r \in R \setminus (L :_R M)$ such that $\mathfrak{q}_1 = ((L :_R M) : r)$ and therefore $\mathfrak{q}_1 rM \subseteq L$. Since $r \notin (L :_R M)$, it follows that there exists an element $x \in M$ such that

$rx \notin L$. Now there exists $N \in E$ such that $N \subseteq L + Rrx$. In particular,

$$\mathfrak{q}_1 N \subseteq L + \mathfrak{q}_1 rx \subseteq L + \mathfrak{q}_1 rM \subseteq L.$$

Since $\mathfrak{q}_1 N$ is finitely generated, so $|V(\mathfrak{q}_1 N) \cap \mathrm{Min}(M)| = \infty$. But in this case for all $\mathfrak{P} \in (V(\mathfrak{q}_1 N) \cap \mathrm{Min}(M)) \backslash (V(N) \cap \mathrm{Min}(M))$, we have $\mathfrak{q}_1 N \subseteq \mathfrak{P}$ and $N \nsubseteq \mathfrak{P}$. Now if $\mathfrak{P}$ be a $\mathfrak{p}$-Prime submodule, then $\mathfrak{q}_1 \subseteq \mathfrak{p}$ and so $|V(\mathfrak{q}_1) \cap A| = \infty$. Hence $|V(\mathfrak{q}_1 M) \cap \mathrm{Min}(M)| = \infty$. So for all $N \in E$, we have $N \nsubseteq \mathfrak{q}_1 M$ and therefore $\mathfrak{q}_1 M \in F$. Let

$$U := \{\mathfrak{q} \in V(\mathfrak{q}_1) \ : \ \mathfrak{q}M \in F\}.$$

Since R is Noetherian it follows that U has a maximal element, say $\mathfrak{q}_2$. But $\mathfrak{q}_2 M \subseteq H$, for some maximal element H of F. We claim that $(H :_R M)$ is a prime ideal of R. If not, according to the above argument, there exists $\mathfrak{q}_3 \in \mathrm{Ass}_R(R/(H :_R M))$ such that $\mathfrak{q}_3 M \in F$ and $\mathfrak{q}_2 \subseteq (H :_R M) \subseteq \mathfrak{q}_3$. By choosing $\mathfrak{q}_2$, we must have $\mathfrak{q}_2 = \mathfrak{q}_3$, which is a contradiction. Therefore $(H :_R M) = \mathfrak{q}_2$ is a prime ideal. Now we show that H is a $\mathfrak{q}_2$-prime submodule. Otherwise there exist $x \in M \backslash H$ and $r \in R \backslash \mathfrak{q}_2$, such that $rx \in H$. So $r \in Z_R(M/H) = \bigcup_{\mathfrak{q} \in \mathrm{Ass}_R(M/H)} \mathfrak{q}$ and hence there exists $\mathfrak{q}' \in \mathrm{Ass}_R(M/H)$ such that $r \in \mathfrak{q}'$. Consequently, $\mathfrak{q}_2 \subset \mathfrak{q}'$. On the other hand by the definition $\mathfrak{q}' = (H :_R y)$ for some $y \in M \backslash H$. Since $H \subset H + Ry$, it follows that there exists $N \in E$ such that $N \subseteq H + Ry$ and so $\mathfrak{q}' N \subseteq H$. According to the above argument, $|V(\mathfrak{q}' M) \cap \mathrm{Min}(M)| = \infty$ which implies $\mathfrak{q}' M \in F$. Finally, we have $\mathfrak{q}_2 = (H :_R M) \subset \mathfrak{q}'$, which is a contradiction with the choosing $\mathfrak{q}_2$. Therefore H is a $\mathfrak{q}_2$-prime submodule of M. Whence, H contains a minimal prime submodule of M such as $\mathfrak{P}$. By assumption there exists a submodule $N_{\mathfrak{P}}$ of $\mathfrak{P}$ such that $N_{\mathfrak{P}} \subseteq \mathfrak{P} \subseteq H$ and $N_{\mathfrak{P}} \in E$, which is a contradiction. Therefore, $\mathrm{Min}(M)$ is a finite set.

Now the proof of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is complete.

(1) $\Rightarrow$ (4) Follows from Lemma 3.1 and Proposition 2.1.

(4) $\Rightarrow$ (1) $\Leftrightarrow$ (5) Since (5) $\Leftrightarrow$ (4) $\Rightarrow$ (3) is clear so we have (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Now we have the following: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(4) $\Rightarrow$ (6) Is clear.

(6) $\Rightarrow$ (3) Since for every $\mathfrak{P} \in D(0)$, $\mathfrak{P} \nsubseteq \bigcup_{L \in \mathrm{Min}(M) \backslash \{\mathfrak{P}\}} L$, it follows that there exists $x_{\mathfrak{P}} \in \mathfrak{P}$ such that$V(Rx_{\mathfrak{P}}) \cap \mathrm{Min}(M) = \{\mathfrak{P}\}$. On the other hand for all $\mathfrak{P} \in (\mathrm{Min}(M) \backslash D(0))$, we have $V(\mathfrak{P}) \cap \mathrm{Min}\,(M) = \{\mathfrak{P}\}$, where $\mathfrak{P}$ is finitely generated. So the assertion follows.

(6) $\Leftrightarrow$ (7) and (1) $\Rightarrow$ (8), (9) are clear.

(8), (9) $\Rightarrow$ (3) Follow by a similar arguments as in (6) $\Rightarrow$ (3).$\square$

The following results follow from Theorem 3.4.

Corollary 3.5. *Let R be a Noetherian ring, M an R-module and B be a proper submodule of M. Then* $\mathrm{Min}(B)$ *is infinite if and only if there exists* $\mathfrak{P} \in D(B)$ *such that* $\mathfrak{P} \subseteq \bigcup_{L \in (\mathrm{Min}(B) \setminus \{\mathfrak{P}\})} L$.

Proof. Follows immediately from Theorem 3.4.□

Corollary 3.6. *Let R be a Noetherian ring, M an R-module and B be a proper submodule of M such that any minimal prime submodule over B is finitely generated. Then* $\mathrm{Min}(B)$ *is finite.*

Proof. Follows immediately from Theorem 3.4.□

Definition 3.7. Let R be a Noetherian ring, $M \neq 0$ be a finitely generated R-module and N be a proper submodule of M. Then *the radical of* N is defined as:

$$\mathrm{Rad}(N) = \bigcap_{L \in \mathrm{Min}\, N} L.$$

Before bringing the next definition, recall that for any ideal I of a Noetherian ring, the *arithmetic rank* of I, denoted by $\mathrm{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I, i.e.,

$$\mathrm{ara}(I) := \min\{n \in \mathbb{N}_0 : \exists\, x_1, \ldots, x_n \in I \,\text{with}\, \mathrm{Rad}((x_1, \ldots, x_n)) = \mathrm{Rad}(I)\}.$$

Definition 3.8. Let R be a Noetherian ring, $M \neq 0$ be a finitely generated R-module and N be a proper submodule of M. We define the *arithmetic rank* of N, as:

$$\mathrm{ara}(N) := \min\{n \in \mathbb{N}_0 : \exists\, x_1, \ldots, x_n \in N \,\text{with}\, \mathrm{Rad}((x_1, \ldots, x_n)) = \mathrm{Rad}(N)\}.$$

The next theorem is a generalization of [15, Theorem 2.7].

Theorem 3.9. *Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module and N be a proper submodule of M. Then* $\mathrm{ara}(N) \leq \mathrm{dimSpec}_R(M) + 1$

Proof. Let $d := \mathrm{dimSpec}_R(M)$. We may assume that d is finite. Now, suppose, to the contrary, that $\mathrm{ara}(N) > d + 1$. Let $n := \mathrm{ara}(N)$. Since $n > d + 1 \geq 1$ it follows from the definition that there exist elements $x_1, \ldots, x_n$ in N such that $\mathrm{Rad}(N) = \mathrm{Rad}((x_1, \ldots, x_n))$. As $n > 0$ it follows that $\mathrm{Min}(0) \setminus V(N) \neq \emptyset$. Therefore it follows from Lemma 3.1 and proposition 2.1 that

$N \nsubseteq \bigcup_{L\in \mathrm{Min}(0)\setminus V(N)} L$. Therefore $(x_1, \ldots, x_n) \nsubseteq \bigcup_{L\in \mathrm{Min}(0)\setminus V(N)} L$, and so by Proposition 2.2 there is $a_1 \in (x_2, \ldots, x_n)$ such that

$$x_1 + a_1 \notin \bigcup_{L\in \mathrm{Min}(0)\setminus V(N)} L.$$

Let $y_1 := x_1 + a_1$. Then $y_1 \in N$ and $\mathrm{Rad}(N) = \mathrm{Rad}((y_1, x_2, \ldots, x_n))$. We shall construct the sequence $y_1, \ldots, y_{n-1} \in N$ such that $\mathrm{Rad}(N) = \mathrm{Rad}((y_1, \ldots, y_{n-1}, x_n))$ and $y_j \notin \bigcup_{L\in \mathrm{Min}((y_1,\ldots,y_{j-1})\setminus V(N)} L$, for each $1 \le j \le n-1$, by an inductive process. To do this end, assume that $1 \le k < n-1$, and that we have already constructed elements $y_1, \ldots, y_k$ such that

$$\mathrm{Rad}(N) = \mathrm{Rad}((y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)).$$

We show how to construct y_{k+1}. To do this, as $k < n-1$ it follows that

$$\mathrm{Min}((y_1, \ldots, y_k))\setminus V(N) \neq \emptyset.$$

Therefore it follows from Lemma 3.1 and proposition 2.1 that

$$N \nsubseteq \bigcup_{L\in \mathrm{Min}((y_1,\ldots,y_k))\setminus V(N)} L.$$

Therefore $(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \nsubseteq \bigcup_{L\in \mathrm{Min}((y_1,\ldots,y_k))\setminus V(N)} L$, and so by Proposition 2.2 there is $a_{k+1} \in (y_1, \ldots, y_k, x_{k+2}, \ldots, x_n)$ such that

$$x_{k+1} + a_{k+1} \notin \bigcup_{L\in \mathrm{Min}((y_1,\ldots,y_k))\setminus V(N)} L.$$

Let $y_{k+1} := x_{k+1} + a_{k+1}$. Then $y_{k+1} \in N$ and $\mathrm{Rad}(N) = \mathrm{Rad}((y_1, \ldots, y_k, y_{k+1}, x_{k+2} \ldots, x_n))$. This completes the inductive step in the construction. Now it is easy to see that $\mathrm{Min}((y_1, \ldots, y_{n-1}))\setminus V(N) \neq \emptyset$. Also, by using an induction argument we can deduce that for any $1 \le j \le n-1$ and any $L \in \mathrm{Min}((y_1, \ldots, y_j))\setminus V(N)$ we have $\mathrm{ht}(L) \ge j$. Consequently, since there exists a prime submodule L of M in which $L \in \mathrm{Min}((y_1, \ldots, y_{n-1}))\setminus V(N)$ it follows that $n-1 \le \mathrm{ht}(L) \le \mathrm{dimSpec}_R(M) = d$, which implies that $n \le d+1$, as required.□

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References

[1] S. Abu-Saymeh, *On dimensions of finitely generated modules*, Comm. Alg. **23**(1995), 1131-1144.

[2] J. Azami and M. Khajepour, *Topics in prime submodules and other aspects of the prime avoidence theorem*, Preprint(Mathematical Reports) .

[3] K. Bahmanpour, A. Khojali and R. Naghipour, *A note on minimal prime divisors of an ideal*, Algebra Colloq. **18**(2011), 727-732.

[4] M. Behboodi, *A generalization of the classical Krull dimension for modules*, J. Algebra. **305**(2006), 1128-1148.

[5] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Univ. Press, Cambridge, UK, 1993.

[6] J. Dauns, *Prime modules*, J. Reine Anegew. Math. **298**(1978), 156-181.

[7] J. Jenkins and P. F. Smith, *On the prime radical of a module over commutative ring*, Comm. Alg. **20**(1992), 3593-3602.

[8] O. A. S. Karamzadeh, *The Prime Avoidance Lemma revisited*, Kyungpook Math. J. **52**(2012), 149-153.

[9] C. P. Lu, *Unions of prime submodules*, Houston J. Math. **23**, no.2(1997), 203-213.

[10] K. H. Leung and S. H. Man, *On commutative Noetherian rings which satisfy the radical formula*, Glasgow math. J. **39**(1997), 285-293.

[11] S. H. Man and P. F. Smith, *On chains of prime submodules*, Israel J. Math. **127**(2002), 131-155.

[12] A. Marcelo and J. Munoz Maque, *Prime submodules, the discent invariant, and modules of finite length*, J. Algebra **189**(1997), 273-293.

[13] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, UK, 1986.

[14] R. L. McCasland and P. F. Smith, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math. **23**(1993), 1041-1062.

[15] A. A. Mehrvarz, K. Bahmanpour and R. Naghipour, *Arithmetic rank, cohomological dimension and filter regular sequences*, J. Alg. Appl. **8**(2009), 855-862.

[16] J. J. Rotman, *An introduction to homological algebra*, Pure Appl. Math., Academic Press, New York, 1979.

[17] D. Pusat-Yilmaz and P. F. Smith, *Chain conditions in modules with krull dimension*, Comm. Alg. **24(13)**(1996), 4123-4133.

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