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APPLICATIONS OF A GROUP IN GENERAL FUZZY AUTOMATA

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ABSTRACT. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and the set of its states be a group. The aim of this paper is the study of applications of a group in a general fuzzy automaton. For this purpose, we define the concepts of fuzzy normal kernel of a general fuzzy automaton, fuzzy kernel of a general fuzzy automaton, adjustable, multiplicative. Then we obtain the relationships between them.

1. Introduction

The concept of fuzzy automata was introduced by Wee in 1967 [11].
Let Σ be a set. A word in Σ is the product of a finite sequence of elements in Σ. Λ will denote the empty word and Σ^* the set of all words on Σ. The length $\ell(x)$ of the word $x \in \Sigma^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\widetilde{P}(X)$ will denote the set of all fuzzy sets on X and $P(X)$ will denote the set of all subsets on X.
A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the total function f from $Q \times \Sigma$ into

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Q is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state. A word $x = x_1x_2 \ldots x_n \in \Sigma^*$ is said to be accepted by A if there exist states $q_0, q_1, \ldots, q_n$ satisfying

(1) $q_0 = s$

(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \ldots, n$,

(3)$q_n \in T$.

The empty word is accepted by A if and only if $s \in T$.

A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, the partial function f from $Q \times \Sigma$ into $P(Q)$ is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A fuzzy finite-state automaton (FFA) is a six-tuple $\widetilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the initial state of $\widetilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition, there is a membership value in $[0, 1]$ called the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition. Otherwise, it specifies the transition itself. The set of all transitions of $\widetilde{F}$ will be denoted by Δ. The above definition is generally accepted as a formal definition of a fuzzy finite-state automaton [4, 5, 6, 7, 8, 9].

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [1].

In this paper, we define the concepts of fuzzy normal kernel of a general fuzzy automaton, fuzzy kernel of a general fuzzy automaton, adjustable, multiplicative and obtain the relationships between them.

Definition 1.1. [1] A general fuzzy automaton (GFA) is an eight-tuple machine $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \widetilde{\delta}, \omega, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,

(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,

(iii) $\widetilde{R}$ is the set of fuzzy start states, $\widetilde{R} \in \widetilde{P}(Q)$,

(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : Q \to Z$ is the output function,

(vi) $\widetilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function,

(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,
(viii) $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition and $\delta : Q \times \Sigma \times Q \to [0,1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0,1]$. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \widetilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that $t \in \{0, 1, 2, , n\}$ and the membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the fuzzy set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \widetilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1. *[1] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:*

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i.

$$v_i = \widetilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

where

- n *is the number of simultaneous transitions to the active state* q_j *at time* $t+1$.
- $\delta(q_i, a_k, q_j)$ *is the weight of a transition from* q_i *to* q_j *upon input* a_k.
- $\mu^t(q_i)$ *is the membership value of* q_i *at time* t.
- $\mu^{t+1}(q_j)$ *is the final membership value of* q_j *at time* $t+1$.

Definition 1.2. [12] Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ such that :

$$\widetilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0,1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\widetilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

Also, if the input at time t_i be u_i, where $u_i \in \Sigma, \forall 1 \leq i \leq n$, then
$\widetilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \widetilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,
$\widetilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\widetilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \widetilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$,
and recursively
$\widetilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\widetilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \widetilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$
$\wedge \widetilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\}$.

If $q \in Q_{act}(t_i)$, we should write q belongs to an element of Q_{act}. Hereafter, we simply denote it as: $q \in Q_{act}$.

Definition 1.3. [5] A fuzzy subset μ of X is a function of X into the closed interval [0,1]. The support of μ is defined to be the set,

$$Supp(\mu) = \{x \in X | \mu(x) > 0\}$$

Definition 1.4. [5] Let λ and μ be fuzzy subsets of G. The product $\lambda * \mu$ of λ and μ is defined by

$$(\lambda * \mu)(x) = \vee\{\lambda(y) \wedge \mu(z) | y, z \in G, x = y * z\}$$

for all $x \in G$.

We let $FP(X)$ denote the fuzzy power set of X. So $FP(X)$ is the set of all fuzzy subsets of X.

$FP(X)$ is a semigroup with respect to the product of fuzzy subsets of X.

Definition 1.5. [5] Let $(G, *)$ be a group. A fuzzy subset λ of G is called a fuzzy subgroup of G if the following properties hold:
i) $\lambda(x * y) \geq \lambda(x) \wedge \lambda(y)$
ii) $\lambda(x) = \lambda(x^{-1})$
for all $x, y \in G$.

Definition 1.6. [5] A fuzzy subgroup λ of G is called a fuzzy normal subgroup of G if

$$\lambda(x * y * x^{-1}) \geq \lambda(y)$$

for all $x, y \in G$.

Definition 1.7. [5] Let λ and μ be fuzzy subsets of G and $\lambda \subseteq \mu$. Then λ is called a fuzzy normal subgroup of μ if

$$\lambda(x * y * x^{-1}) \geq \lambda(x) \wedge \lambda(y)$$

for all $x, y \in G$.

2. Applications of a group in general fuzzy automata

Definition 2.1. Let $\widetilde{F_1} = (Q_1, \Sigma_1, \widetilde{R}_1, Z, \omega, \widetilde{\delta}_1, F_1, F_2)$ and $\widetilde{F_2} = (Q_2, \Sigma_2, \widetilde{R}_2, Z, \omega, \widetilde{\delta}_2, F_1, F_2)$ be general fuzzy automata and $(Q_1, *)$ and $(Q_2, *)$ be groups. A pair of functions (f, g), where $f : Q_1 \to Q_2$, $g : \Sigma_1 \to \Sigma_2$, is called a homomorphism from $\widetilde{F_1}$ into $\widetilde{F_2}$ if the following conditions hold:
i) f is a group homomorphism,
ii) $\widetilde{\delta}_1((p, \mu^t(p)), x, q) \leq \widetilde{\delta}_2((f(p), \mu^t(f(p))), g(x), f(q))$, $\forall p, q \in Q, x \in \Sigma_1$.

Definition 2.2. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and let λ be a fuzzy subset of Q and $(Q, *)$ be a group. λ is called fuzzy normal kernel of $\widetilde{F}$ if the following conditions hold:
i) λ is a fuzzy normal subgroup of Q,
ii) $\lambda(p * r^{-1}) \geq \widetilde{\delta}((q * k, \mu^t(q * k)), x, p) \wedge \widetilde{\delta}((q, \mu^t(q)), x, r) \wedge \lambda(k)$
for all $p, q, k, r \in Q$, $x \in \Sigma$.

Theorem 2.3. *Let $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and let λ is a fuzzy normal subgroup of Q and $(Q, *)$ be a group. Then λ is a fuzzy normal kernel of $\widetilde{F}^*$ if and only if*

$$\lambda(p * r^{-1}) \geq \widetilde{\delta}^*((q * k, \mu^t(q * k)), x, p) \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, r) \wedge \lambda(k)$$

for all $p, q, k, r \in Q$, $x \in \Sigma^*$.

Proof. Let λ be a fuzzy normal kernel of $\widetilde{F}^*$. We prove the theorem by induction on $|x| = n$. Let $n = 0$. Then $x = \Lambda$. If $p = q * k$, $r = q$, then since λ is a fuzzy normal subgroup, we have

$$\widetilde{\delta}^*((q*k, \mu^t(q*k)), x, p) \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, r) \wedge \lambda(k) = \lambda(k) \leq \lambda(q*k*q^{-1}) = \lambda(p*r^{-1})$$

If $p \neq q * k$ or $r \neq q$, then

$$\widetilde{\delta}^*((q*k, \mu^t(q*k)), x, p) \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, r) \wedge \lambda(k) = 0 \leq \lambda(p*r^{-1})$$

Thus the result holds for $n = 0$. Suppose that the result holds for all $x \in \Sigma^*$ such that $|x| = n-1$. Let $x = ya$, $y \in \Sigma^*$, $|y| = n-1$, $n > 0$. Then we have

$$\begin{aligned}
&\widetilde{\delta}^*((q*k, \mu^t(q*k)), x, p) \wedge \widetilde{\delta}^*((q, \mu^t(q)), x, r) \wedge \lambda(k) \\
&= \widetilde{\delta}^*((q*k, \mu^t(q*k)), ya, p) \wedge \widetilde{\delta}^*((q, \mu^t(q)), ya, r) \wedge \lambda(k) \\
&= (\vee_{u\in Q}\widetilde{\delta}^*((q*k, \mu^t(q*k)), y, u) \wedge \widetilde{\delta}((u, \mu^t(u)), a, p)) \wedge \\
&\quad (\vee_{v\in Q}\widetilde{\delta}^*((q, \mu^t(q)), y, v) \wedge \widetilde{\delta}((v, \mu^t(v)), a, r)) \wedge \lambda(k) \\
&= \vee_{u\in Q} \vee_{v\in Q} \widetilde{\delta}^*((q*k, \mu^t(q*k)), y, u) \wedge \widetilde{\delta}((u, \mu^t(u)), a, p) \wedge \\
&\quad \widetilde{\delta}^*((q, \mu^t(q)), y, v) \wedge \widetilde{\delta}((v, \mu^t(v)), a, r) \wedge \lambda(k) \\
&\leq \vee_{u\in Q} \vee_{v\in Q} \lambda(u*v^{-1}) \wedge \widetilde{\delta}((u, \mu^t(u)), a, p) \wedge \\
&\quad \widetilde{\delta}((v, \mu^t(v)), a, r) \\
&\leq \vee_{u\in Q} \vee_{v\in Q} \lambda(v^{-1}*u) \wedge \widetilde{\delta}((v*v^{-1}*u, \mu^t(v*v^{-1}*u)), a, p) \wedge \\
&\quad \widetilde{\delta}((v, \mu^t(v)), a, r) \\
&= \vee_{u\in Q} \vee_{v\in Q} \lambda(k') \wedge \widetilde{\delta}((v*k', \mu^t(v*k')), a, p) \wedge \widetilde{\delta}((v, \mu^t(v)), a, r) \\
&\leq \lambda(p*r^{-1})
\end{aligned}$$

The converse is trivial. □

Definition 2.4. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and let λ be a fuzzy subset of Q and $(Q, *)$ be a group. λ is called fuzzy kernel of $\widetilde{F}$ if the following conditions hold:

i)λ is a fuzzy subgroup of Q,

ii)$\lambda(p) \geq \widetilde{\delta}((q, \mu^t(q)), x, p) \wedge \lambda(q)$ for all $p, q \in Q$, $x \in \Sigma$.

Theorem 2.5. *Let $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and let λ is a fuzzy subgroup of Q and $(Q, *)$ be a group. Then λ is a fuzzy kernel of $\widetilde{F}^*$ if and only if $\lambda(p) \geq \widetilde{\delta}^*((q, \mu^t(q)), x, p) \wedge \lambda(q)$ for all $p, q \in Q$, $x \in \Sigma^*$.*

Proof. The proof is similar to that of Theorem 2.3. □

Definition 2.6. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $(Q, *)$ be a group. $\widetilde{F}$ is called adjustable if

$$\widetilde{\delta}((p * q, \mu^t(p * q)), x, p) \leq \widetilde{\delta}((p, \mu^t(p)), x, k)$$

for all $p, q, r, k \in Q$, $x \in \Sigma$.

Theorem 2.7. *Let $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be an adjustable general fuzzy automaton, $(Q, *)$ be a group and let λ be a fuzzy normal kernel of $\widetilde{F}$ and ν be a fuzzy kernel of $\widetilde{F}$. Then $\lambda * \nu$ is a fuzzy kernel of $\widetilde{F}^*$.*

Proof. Since λ is a fuzzy normal subgroup of Q and ν is a fuzzy subgroup of Q, it follows that $\lambda * \nu$ is a fuzzy subgroup of Q and $\lambda * \nu = \nu * \lambda$. Since $p = (p * r^{-1}) * r$, then we have

$$(\lambda * \nu)(p) \geq \lambda(p * r^{-1}) \wedge \nu(r)$$

$$\geq (\widetilde{\delta}((a * b, \mu^t(a * b)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \lambda(b)) \wedge (\widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \nu(a))$$

for all $a, b, p \in Q$, $x \in \Sigma$.
Since $\widetilde{F}$ is adjustable, so we have

$$\widetilde{\delta}((a * b, \mu^t(a * b)), x, p) \leq \widetilde{\delta}((a, \mu^t(a)), x, r)$$

Then we have

$$(\widetilde{\delta}((a * b, \mu^t(a * b)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \lambda(b)) \wedge (\widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \nu(a))$$

$$= \widetilde{\delta}((a * b, \mu^t(a * b)), x, p) \wedge \lambda(b) \wedge \nu(a)$$

Thus for all $p, q \in Q$, $x \in \Sigma$,

$$(\lambda * \nu)(p) \geq \vee\{\widetilde{\delta}((a * b, \mu^t(a * b)), x, p) \wedge \lambda(b) \wedge \nu(a) | a, b \in Q, a * b = q\}$$

$$= \widetilde{\delta}((q, \mu^t(q)), x, p) \wedge (\vee\{\lambda(b) \wedge \nu(a) | a, b \in Q, a * b = q\})$$

$$= \widetilde{\delta}((q, \mu^t(q)), x, p) \wedge (\nu * \lambda)(q)$$

$$= \widetilde{\delta}((q, \mu^t(q)), x, p) \wedge (\lambda * \nu)(q)$$

Hence $\lambda * \nu$ is a fuzzy kernel of $\widetilde{F}$. □

Theorem 2.8. *Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be an adjustable general fuzzy automaton and $(Q, *)$ be a group. If λ and ν are fuzzy normal kernels of $\widetilde{F}$, then $\lambda * \nu$ is a fuzzy normal kernel of $\widetilde{F}$.*

Proof. Since λ and ν are fuzzy normal subgroups of Q, it follows that $\lambda * \nu$ is a fuzzy normal subgroup of Q and $\lambda * \nu = \nu * \lambda$. Since $p * r^{-1} = (p * q^{-1}) * (q * r^{-1})$, then we have

$$(\lambda * \nu)(p * r^{-1}) \geq \lambda(p * q^{-1}) \wedge \nu(q * r^{-1})$$

$$\geq (\widetilde{\delta}((a * b * c, \mu^t(a * b * c)), x, p) \wedge \widetilde{\delta}((a * b, \mu^t(a * b)), x, q) \wedge \lambda(c))$$

$$\wedge(\widetilde{\delta}((a * b, \mu^t(a * b)), x, q) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \nu(b))$$

for all $a, b, c, p, r \in Q$, $x \in \Sigma$.
Since $\widetilde{F}$ is adjustable, so we have

$$\widetilde{\delta}((a * b * c, \mu^t(a * b * c)), x, p) \leq \widetilde{\delta}((a * b, \mu^t(a * b)), x, q)$$

Then we have

$$\widetilde{\delta}((a * b * c, \mu^t(a * b * c)), x, p) \wedge \widetilde{\delta}((a * b, \mu^t(a * b)), x, q) \wedge \lambda(c)$$

$$\wedge\widetilde{\delta}((a * b, \mu^t(a * b)), x, q) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \nu(b)$$

$$= \widetilde{\delta}((a * b * c, \mu^t(a * b * c)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \lambda(c) \wedge \nu(b)$$

Thus for all $p, q, r, k \in Q$, $x \in \Sigma$, we have

$$(\lambda * \nu)(p * r^{-1}) \geq$$

$$\vee\{\widetilde{\delta}((a * b * c, \mu^t(a * b * c)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge \lambda(c) \wedge \nu(b) | b, c \in Q, b * c = k\}$$

$$= \widetilde{\delta}((a * k, \mu^t(a * k)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge (\vee\{\lambda(c) \wedge \nu(b) | b, c \in Q, b * c = k\})$$

$$= \widetilde{\delta}((a * k, \mu^t(a * k)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge (\nu * \lambda)(k)$$

$$= \widetilde{\delta}((a * k, \mu^t(a * k)), x, p) \wedge \widetilde{\delta}((a, \mu^t(a)), x, r) \wedge (\lambda * \nu)(k)$$

Hence $\lambda * \nu$ is a fuzzy normal kernel of $\widetilde{F}$. □

Theorem 2.9. *Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton, $(Q, *)$ be a group and let λ be a fuzzy normal kernel of $\widetilde{F}$ with $Supp(\lambda) = Q$. Then there exists a general fuzzy automaton say $\widetilde{H}$ and there exists a homomorphism from $\widetilde{F}$ to $\widetilde{H}$.*

Proof. Now λ is a fuzzy normal subgroup of Q. Define $\lambda_p : Q \to [0,1]$ by $\lambda_p(q) = \lambda(p * q^{-1})$ for all $q \in Q$. Let $E = \{\lambda_p | p \in Q\}$. Then E is a group with respect to the binary operation $\lambda_p * \lambda_q = \lambda_{p*q}$ for all $\lambda_p, \lambda_q \in E$. Define $Q/\lambda : E \to [0,1]$ by $(Q/\lambda)(\lambda_p) = \lambda(p)$ for all $\lambda_p \in E$. Thus Q/λ is a fuzzy subgroup of E. Let $G = Supp(Q/\lambda)$. Define $\widetilde{\sigma} : (G \times [0,1]) \times \Sigma \times G \to [0,1]$ by

$$\widetilde{\sigma}((\lambda_p, \mu^t(\lambda_p)), x, \lambda_q) = \vee\{\widetilde{\delta}((a, \mu^t(a)), x, b) | a, b \in Q, \lambda_a = \lambda_p, \lambda_b = \lambda_q\}$$

for all $\lambda_p, \lambda_q \in G$. Define $f : Q \to G$ by $f(q) = \lambda_q$ for all $q \in Q$ and let $g : \Sigma \to \Sigma$ be idendity map. Then we have

$$f(p * q) = \lambda_{p*q} = \lambda_p * \lambda_q = f(p) * f(q)$$

for all $p, q \in Q$.
Thus (f, g) is a homomorphism from $\widetilde{F}$ to $\widetilde{H} = (G, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\sigma}, F_1, F_2)$. □

Definition 2.10. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and let λ be a fuzzy kernel of $\widetilde{F}$ and $(Q, *)$ be a group. A fuzzy set ν of Q is called a fuzzy normal kernel of λ if the following conditions hold:
i) $\nu \subseteq \lambda$ and ν is a fuzzy normal subgroup of λ,
ii) $\nu(p * r^{-1}) \geq \widetilde{\delta}((q * k, \mu^t(q * k)), x, p) \wedge \widetilde{\delta}((q, \mu^t(q)), x, r) \wedge \nu(k)$
for all $p, k, r \in Q$, $q \in Supp(\lambda)$, $x \in \Sigma$.

Definition 2.11. Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be a general fuzzy automaton and $(Q, *)$ be a group. $\widetilde{F}$ is called multiplicative if having the following properties:
i) there exists the element x_0 of Σ such that $\widetilde{\delta}((e, \mu^t(e)), x_0, e) > 0$
ii) $\widetilde{\delta}((q, \mu^t(q)), x, p * r) = \widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge \widetilde{\delta}((e, \mu^t(e)), x, r)$,
iii) $\widetilde{\delta}((p_1 * p_2^{-1}, \mu^t(p_1 * p_2^{-1})), x_0, q_1 * q_2^{-1}) = \widetilde{\delta}((p_1, \mu^t(p_1)), x_0, q_1) \wedge \widetilde{\delta}((p_2, \mu^t(p_2)), x_0, q_2)$
for all $p, q, r, p_1, p_2, q_1, q_2 \in Q$, $x \in \Sigma$.

Theorem 2.12. *Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be an adjustable multiplicative general fuzzy automaton and $(Q, *)$ be an abelian group. Let λ be a fuzzy kernel of $\widetilde{F}$, ν be a fuzzy normal kernel of λ and $Supp(\lambda) \neq \Phi$. If ν is a fuzzy normal subgroup of Q, then ν is a fuzzy normal kernel of $\widetilde{F}$.*

Proof. Since $\widetilde{F}$ is multiplicative, for all $p, q, r, k \in Q$, $x \in \Sigma$, we have

$$\begin{aligned}
&\widetilde{\delta}((q * k, \mu^t(q * k)), x, p) \wedge \widetilde{\delta}((q, \mu^t(q)), x, r) \\
&= \widetilde{\delta}((q * k, \mu^t(q * k)), x, p * e) \wedge \widetilde{\delta}((q, \mu^t(q)), x, r * e) \\
&= (\widetilde{\delta}((q * k, \mu^t(q * k)), x_0, p) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge (\widetilde{\delta}((q, \mu^t(q)), x_0, r) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \\
&= (\widetilde{\delta}((q * k, \mu^t(q * k)), x_0, p * e) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge (\widetilde{\delta}((q, \mu^t(q)), x_0, r) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e))
\end{aligned}$$

$$= \widetilde{\delta}((q,\mu^t(q)),x_0,p) \wedge \widetilde{\delta}((k^{-1},\mu^t(k^{-1})),x_0,e) \wedge \widetilde{\delta}((e,\mu^t(e)),x,e) \wedge \widetilde{\delta}((q,\mu^t(q)),x_0,r)$$

Now for any $q \in Q$, $b \in Supp(\lambda)$, $q = b * (q^{-1} * b)^{-1}$. Thus we have

$$\widetilde{\delta}((q*k,\mu^t(q*k)),x,p) \wedge \widetilde{\delta}((q,\mu^t(q)),x,r)$$

$$= \widetilde{\delta}((b*(q^{-1}*b)^{-1}*k,\mu^t(b*(q^{-1}*b)^{-1}*k)),x,p*e) \wedge \widetilde{\delta}((e*q,\mu^t(e*q)),x,r*e)$$

$$= \widetilde{\delta}((b,\mu^t(b)),x_0,p) \wedge \widetilde{\delta}((q^{-1}*b,\mu^t(q^{-1}*b)),x_0,e) \wedge$$

$$\widetilde{\delta}((k^{-1},\mu^t(k^{-1})),x_0,e) \wedge \widetilde{\delta}((e,\mu^t(e)),x,e) \wedge \widetilde{\delta}((q^{-1},\mu^t(q^{-1})),x_0,r)$$

$$= \widetilde{\delta}((b,\mu^t(b)),x_0,p) \wedge \widetilde{\delta}((k^{-1},\mu^t(k^{-1})),x_0,e) \wedge \widetilde{\delta}((e,\mu^t(e)),x,e) \wedge$$

$$(\widetilde{\delta}((q^{-1}*b,\mu^t(q^{-1}*b)),x_0,e) \wedge \widetilde{\delta}((q^{-1},\mu^t(q^{-1})),x_0,r))$$

$$= \widetilde{\delta}((b,\mu^t(b)),x_0,p) \wedge \widetilde{\delta}((k^{-1},\mu^t(k^{-1})),x_0,e) \wedge \widetilde{\delta}((e,\mu^t(e)),x,e) \wedge \widetilde{\delta}((b,\mu^t(b)),x_0,r)$$

Since ν is a fuzzy normal kernel of λ, for all $p,q,r \in Q$, $b \in Supp(\lambda)$, $x \in \Sigma$, we have

$$\nu(p*r^{-1}) \geq \widetilde{\delta}((b*k,\mu^t(b*k)),x,p) \wedge \widetilde{\delta}((b,\mu^t(b)),x,r) \wedge \nu(k)$$

$$= \widetilde{\delta}((b*k,\mu^t(b*k)),x,p*e) \wedge \widetilde{\delta}((b,\mu^t(b)),x,r*e) \wedge \nu(k)$$

$$= \widetilde{\delta}((b,\mu^t(b)),x_0,p) \wedge \widetilde{\delta}((k^{-1},\mu^t(k^{-1})),x_0,e) \wedge$$

$$\widetilde{\delta}((b,\mu^t(b)),x_0,r) \wedge \widetilde{\delta}((e,\mu^t(e)),x,e) \wedge \nu(k)$$

$$= \widetilde{\delta}((q*k,\mu^t(q*k)),x,p) \wedge \widetilde{\delta}((q,\mu^t(q)),x,r) \wedge \nu(k)$$

Hence ν is a fuzzy normal kernel of $\widetilde{F}$. □

Theorem 2.13. *Let $\widetilde{F} = (Q,\Sigma,\widetilde{R},Z,\omega,\widetilde{\delta},F_1,F_2)$ be an adjustable multiplicative general fuzzy automaton, $(Q,*)$ be a group, λ be a fuzzy subset of Q and $Supp(\lambda) \neq \Phi$. Then the following statements are equivalent:*
i) λ is a fuzzy normal kernel of $\widetilde{F}$
ii) λ is a fuzzy normal subgroup of Q and $\lambda(q) \geq \widetilde{\delta}((p,\mu^t(p)),x_0,q) \wedge \lambda(p)$, for all $p,q \in Q$.

Proof. $(i \Rightarrow ii)$ Since λ is a fuzzy normal kernel of $\widetilde{F}$, we have:

$$\lambda(q) = \lambda(q*e^{-1})$$

$$\geq \widetilde{\delta}((e*p,\mu^t(e*p)),x_0,q) \wedge \widetilde{\delta}((e,\mu^t(e)),x_0,e) \wedge \lambda(p)$$

$$= \widetilde{\delta}((p,\mu^t(p)),x_0,q) \wedge \lambda(p)$$

(Since $\widetilde{F}$ is adjustable, for all $p,q \in Q$, we have
$\widetilde{\delta}((p,\mu^t(p)),x_0,q) = \widetilde{\delta}((e*p,\mu^t(e*p)),x_0,q) \leq \widetilde{\delta}((e,\mu^t(e)),x_0,e)$)

$(ii \Rightarrow i)$ Since $\widetilde{F}$ is multiplicative, for all $p, q, r, k \in Q$, $x \in \Sigma$, we have

$$\widetilde{\delta}((q*k, \mu^t(q*k)), x, p*r) \wedge \widetilde{\delta}((q, \mu^t(q)), x, p) \wedge \lambda(k)$$

$$= \widetilde{\delta}((q*k, \mu^t(q*k)), x, p*r*e) \wedge \widetilde{\delta}((q, \mu^t(q)), x, p*e) \wedge \lambda(k)$$

$$= (\widetilde{\delta}((q*k, \mu^t(q*k)), x_0, p*r) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge$$

$$(\widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge \lambda(k)$$

$$= \widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge \widetilde{\delta}((k^{-1}, \mu^t(k^{-1})), x_0, r^{-1}) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge \lambda(k)$$

$$= \widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge \widetilde{\delta}((e \star k^{-1}, \mu^t(e*k^{-1})), x_0, e*r^{-1}) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge \lambda(k)$$

$$= \widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge$$

$$(\widetilde{\delta}((e, \mu^t(e)), x_0, e) \wedge \widetilde{\delta}((k, \mu^t(k)), x_0, r))$$

$$\wedge \widetilde{\delta}((e, \mu^t(e)), x, e)) \wedge \lambda(k)$$

$$\leq \widetilde{\delta}((k, \mu^t(k)), x_0, r) \wedge \lambda(k)$$

$$\leq \lambda(r) \leq \lambda(p*r*p^{-1})$$

Now let $p_1, p_2, q, k \in Q$, $x \in \Sigma$. Since Q is a group, then there exists a unique element $r \in Q$ such that $p_1 = p_2 * r$. Thus

$$\lambda(p_1 * p_2^{-1}) = \lambda(p_2 * r * p_2^{-1})$$

$$\geq \widetilde{\delta}((q*k, \mu^t(q*k)), x, p_2*r) \wedge \widetilde{\delta}((q, \mu^t(q)), x, p_2) \wedge \lambda(k)$$

$$= \widetilde{\delta}((q*k, \mu^t(q*k)), x, p_1) \wedge \widetilde{\delta}((q, \mu^t(q)), x, p_2) \wedge \lambda(k)$$

Hence λ is a fuzzy normal kernel of $\widetilde{F}$. □

Theorem 2.14. *Let $\widetilde{F} = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}, F_1, F_2)$ be an adjustable multiplicative general fuzzy automaton and $(Q, *)$ be a group. Then there exists a semigroup homomorphism $f : Q \to FP(Q)$ and there exists a function $g : \Sigma \to FP(Q)$ such that $\widetilde{\delta}((q, \mu^t(q)), x, p) = f(q)(p) \wedge g(x)(e)$, for all $p, q \in Q$, $x \in \Sigma$.*

Proof. Define $f : Q \to FP(Q)$ by $f(q) = \mu_q$ for all $q \in Q$, where $\mu_q : Q \to [0,1]$ is defined by $\mu_q(p) = \widetilde{\delta}((q, \mu^t(q)), x_0, p)$ for all $p \in Q$. Since $\widetilde{F}$ is multiplicative, for all $p_1, p_2, q_1, q_2 \in Q$, we have

$$\mu_{q_1 * q_2}(p_1 * p_2) = \widetilde{\delta}((q_1 * q_2, \mu^t(q_1 * q_2)), x_0, p_1 * p_2)$$

$$= \widetilde{\delta}((q_1, \mu^t(q_1)), x_0, p_1) \wedge \widetilde{\delta}((q_2^{-1}, \mu^t(q_2^{-1})), x_0, p_2^{-1})$$

$$= \widetilde{\delta}((q_1, \mu^t(q_1)), x_0, p_1) \wedge \widetilde{\delta}((q_2, \mu^t(q_2)), x_0, p_2)$$

$$= \mu_{q_1}(p_1) \wedge \mu_{q_2}(p_2)$$

Also we have

$$(\mu_{q_1} * \mu_{q_2})(p) = \vee\{\mu_{q_1}(q) \wedge \mu_{q_2}(r) | q, r \in Q, p = q * r\} = \mu_{q_1 * q_2}(p)$$

for all $p \in Q$. Hence

$$f(q_1 * q_2) = f(q_1) * f(q_2)$$

For $x \in \Sigma$, define the fuzzy subset λ_x of Q by $\lambda_x(q) = \widetilde{\delta}((e, \mu^t(e)), x, q)$ for all $q \in Q$. Define $g : \Sigma \to FP(Q)$ by $g(x) = \lambda_x$ for all $x \in \Sigma$. Since $\widetilde{F}$ is multiplicative, for all $p, q \in Q$, $x \in \Sigma$, we have

$$f(q)(p) \wedge g(x)(e)$$

$$= \mu_q(p) \wedge \lambda_x(e) = \widetilde{\delta}((q, \mu^t(q)), x_0, p) \wedge \widetilde{\delta}((e, \mu^t(e)), x, e) = \widetilde{\delta}((q, \mu^t(q)), x, p)$$

□

3. **Conclusion**

In this paper, we defined the concepts of fuzzy normal kernel of a general fuzzy automaton, fuzzy kernel of a general fuzzy automaton, adjustable, multiplicative. Then we obtained the relationships between them.

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