NOTE ON REGULAR AND COREGULAR SEQUENCES

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Abstract. Let \( R \) be a commutative Noetherian ring and let \( M \) be a finitely generated \( R \)-module. If \( I \) is an ideal of \( R \) generated by \( M \)-regular sequence, then we study the vanishing of the first Tor functors. Moreover, for Artinian modules and coregular sequences we examine the vanishing of the first Ext functors.

1. Introduction

Throughout this note, we shall assume that \( R \) is a commutative Noetherian ring with non-zero identity and \( I \) an ideal of \( R \). Suppose \( M \) and \( A \) two non-zero finitely generated and Artinian \( R \)-modules, respectively. An ordered sequence \( x_1, \ldots, x_t \) of elements of \( R \) is called an \( M \)-regular sequence [\[, Definition 1.1.1]; if \( x_i \) is not a zero-divisor in \( M/(x_1, \ldots, x_{i-1})M \) for \( i = 1, \ldots, t \), and \( M \neq (x_1, \ldots, x_t)M \). Also, an ordered sequence \( x_1, \ldots, x_i \) of elements of \( R \) is said an \( A \)-coregular sequence [\[, Definition 3.1]; if \( x_i \) is an \((0 :_A (x_1, \ldots, x_{i-1}))\)-coregular element for \( i = 1, \ldots, t \), and \((0 :_A (x_1, \ldots, x_t)) \neq 0 \). Note that an element \( x \) of \( R \) is \( A \)-coregular if \( A = xA \), (see [\[, Definition 2.4]).

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It is known that if $I$ is generated by an $M$-regular sequence, then $\text{Tor}_1^R(R/I, M) = 0$. In particular, if $x_1, ..., x_t$ is an $R$-regular sequence, $I = (x_1, ..., x_s)$ and $J = (x_{s+1}, ..., x_t)$ two ideals of $R$ then $\text{Tor}_1^R(R/I, R/J) = 0$ and so $I \cap J = IJ$.

Our main aim in this article is to extend Proposition 1.1.4 of [1] for powers an ideal that generated by regular sequence and use it over Tor functors. Additionally, we examine the problem on the self-duality.

2. The results

The interplay between regular sequences and homological invariants is a major theme, and numerous arguments will be based on the following proposition which is appear in [1].

Proposition 2.1. Let $R$ be a ring, $M$ an $R$-module, and $x$ an $M$-regular sequence. Then an exact sequence

\[ N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \rightarrow 0, \]

induces an exact sequence

\[ \frac{N_2}{xN_2} \rightarrow \frac{N_1}{xN_1} \rightarrow \frac{N_0}{xN_0} \rightarrow \frac{M}{xM} \rightarrow 0. \]

The following result is a generalization of Proposition 2.1.

Theorem 2.2. Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module, and $I = (a_1, ..., a_t)$ an ideal generated by an $M$-regular sequence. Then an exact sequence

\[ N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \rightarrow 0, \]

of $R$-modules induces an exact sequence

\[ \frac{N_2}{I^n N_2} \rightarrow \frac{N_1}{I^n N_1} \rightarrow \frac{N_0}{I^n N_0} \rightarrow \frac{M}{I^n M} \rightarrow 0, \]

for all $n \geq 1$.

Proof. Since tensor product is a right exact functor, we only need to verify exactness at $\frac{N_2}{I^n N_2}$. Let $\overline{y}$ denote residue classes modulo $I^n N_1$. Let $y_1 \in N_1$ such that $\overline{f_1(y)} = 0$. Then $f_1(y) \in I^n N_0$. Hence we can consider a homogeneous polynomial of degree $n$, $f_1(y) \in N_0[x_1, ..., x_t]$ such that

\[ f_1(y) = \sum_{\alpha_{i_1} + \cdots + \alpha_{i_t} = n} \alpha_{i_1} \cdots \alpha_{i_t} x_1^{\alpha_{i_1}} \cdots x_t^{\alpha_{i_t}} \quad (\ast) \]

for some $\alpha_{i_1} \cdots \alpha_{i_t} \in N_0$. Since

\[ f_0(\alpha_{i_1} \cdots \alpha_{i_t}) x_1^{\alpha_{i_1}} \cdots x_t^{\alpha_{i_t}} = f_0 \circ f_1(y) = 0 \in I^{n+1} M, \quad (\ast\ast) \]

by [1, Theorem 1.1.7] or [2, Theorem 16.2] the coefficients $f_0(\alpha_{i_1} \cdots \alpha_{i_t}) \in IM$, and so $f_0(\alpha_{i_1} \cdots \alpha_{i_t}) = \sum_{j=1}^t m_{ij} x_j$ for some $m_{ij} \in M$. By replacing $m_{ij}$ in the equation (\ast\ast) and
repeating this argument, we have $f_0(\alpha_{i_1 \ldots i_t}) \in I^nM$ for all $n \in \mathbb{N}$. By Krull intersection Theorem, $f_0(\alpha_{i_1 \ldots i_t}) = 0$ and by our hypothesis $\alpha_{i_1 \ldots i_t} \in \text{im}f_1$. Hence there exists $\beta_{i_1 \ldots i_t} \in N_1$ such that $\alpha_{i_1 \ldots i_t} = f_1(\beta_{i_1 \ldots i_t})$. Therefore by replacing this value in the equation $(\ast)$ and our hypothesis, there exists $z \in N_2$ such that $y - \sum_{\alpha_{i_1} + \ldots + \alpha_{i_t} = n} \beta_{i_1 \ldots i_t} x_1^{\alpha_{i_1}} \ldots x_t^{\alpha_{i_t}} = f_2(z)$. Hence $\overline{y} = \overline{f_2(z)}$, as desired. □

**Corollary 2.3.** Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module and $x_1, \ldots, x_s, x_{s+1}, \ldots, x_t$ an $M$-regular sequence. Then the following statements hold:

(i) If $I = (x_1, \ldots, x_s)$, then $\text{Tor}^1(R/I^n, M) = 0$ for all $n \geq 1$.

(ii) If $J = (x_{s+1}, \ldots, x_t)$, then $\text{Tor}^1(R/J^n, M/I^nM) = 0$ for all $m, n \geq 1$.

**Proof.** (i) From the exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, where $F$ is a free $R$-module and by using Theorem 2.1, we obtain the result.

(ii) It is known that $\text{Ass}(M/I^nM) = \text{Ass}(M/IM)$. It therefore follows

$\text{Ass}(M/I^nM + (x_{s+1}, \ldots, x_t)M) = \text{Ass}(M/IM + (x_{s+1}, \ldots, x_t)M)$ for all $n \in \mathbb{N}$ and $s + 1 \leq i \leq t$. Hence $x_{s+1}, \ldots, x_t$ is an $M/I^nM$-regular sequence for all $n \in \mathbb{N}$ and so by Theorem 2.2, the result follows. □

**Corollary 2.4.** Let $(R, m)$ be a local ring and $J \subseteq I$ be two ideals of $R$ such that $I/J$ is generated by $R/J$-regular sequence. Then $I^n \cap J = I^{n-1}J$ for all $n \in \mathbb{N}$.

**Proof.** From the exact sequence

$$0 \rightarrow mI + J/mI \rightarrow I/mI \rightarrow I/mI + J \rightarrow 0,$$

and by using Nakayama’s Lemma, There exists $x_1, \ldots, x_s, y_1, \ldots, y_t$ of $R$ such that $I = (x_1, \ldots, x_s, y_1, \ldots, y_t)$, $J = (x_1, \ldots, x_s)$ and $I/J = (\overline{y_1}, \ldots, \overline{y_t})$, where $y_1, \ldots, y_t$ is an $R/J$-regular sequence. By Corollary 2.3(i) $\text{Tor}^1(R/(y_1, \ldots, y_t)^n, R/J) = 0$ for all $n \in \mathbb{N}$ and so $J \cap (y_1, \ldots, y_t)^n = J(y_1, \ldots, y_t)^n$. Therefore $J \cap I^n = J \cap (J^n + J^{n-1}(y_1, \ldots, y_t) + \ldots + J(y_1, \ldots, y_t)^{n-1} + (y_1, \ldots, y_t)^n) = J^n + J^{n-1}(y_1, \ldots, y_t) + \ldots + J(y_1, \ldots, y_t)^{n-1} + J \cap (y_1, \ldots, y_t)^n = JI^{n-1}$. □

**Proposition 2.5.** Let $M$ be a finitely generated $R$-module and $I = (a_1, \ldots, a_t)$ an ideal generated by an $M$-regular sequence and an $R$-regular sequence. Then $\text{Tor}^1(R/I^n, M) = 0$ for all $n \geq 1$ and all $i \geq 1$.  

Proof. We use induction on $n$. If $n = 1$, then $\text{Tor}_i^R(R/I, M) = 0$ for all $i \geq 1$, by Theorem 2.2. Let $n > 1$ and suppose that the proposition holds for all integer smaller than $n$. By using the exact sequence

$$0 \longrightarrow \frac{I^{n-1}}{I^n} \longrightarrow \frac{R}{I^n} \longrightarrow \frac{R}{I^{n-1}} \longrightarrow 0,$$

and the inductive hypothesis, $\text{Tor}_i^R(R/I, M) \cong \text{Tor}_i^R(\frac{I^{n-1}}{I^n}, M)$ for all $i \geq 1$. Hence, by using [3, Theorem 1.1.8] we have the isomorphism $\frac{I^{n-1}}{I^n} \cong \bigoplus_{j=1}^{(n+t-2)} R/I$. Therefore, we obtain $\text{Tor}_i^R(R/I, M) = 0$ for all $n \geq 1$ and all $i \geq 1$. This completes the inductive step. \[\square\]

Corollary 2.6. Let $(R, m)$ be a Cohen-Macaulay local ring, $M$ a finitely generated maximal Cohen-Macaulay $R$-module, and $I = (a_1, ... , a_t)$ an ideal generated by an $R$-regular sequence. Then $\text{Tor}_i^R(R/I, M) = 0$ for all $i \geq 1$ and all $n \geq 1$.

Proof. Since every $R$-regular sequence is an $M$-regular sequence, the result follows by Proposition 2.5. \[\square\]

The following result is a dual of [3, Proposition 1.1.4].

Proposition 2.7. Let $A$ be an Artinian $R$-module and $I = (a_1, ... , a_t)$ an ideal generated by an $A$-coregular sequence. Then an exact sequence

$$0 \longrightarrow A \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2$$

of $R$-modules induces an exact sequence

$$0 \longrightarrow (0 :_A I) \xrightarrow{g_0} (0 :_{B_0} I) \xrightarrow{g_1} (0 :_{B_1} I) \xrightarrow{g_2} (0 :_{B_2} I).$$

Proof. By induction, it is enough to consider the case in which $I$ consists of a single $A$-coregular element $a = a_1$. We obtain the induced sequence if we use the functor $\text{Hom}_R(\frac{R}{I}, -)$. Since $\text{Hom}_R(\frac{R}{I}, -)$ is a left exact functor, we only need to verify exactness at $(0 :_{B_1} a)$. Let $g| I$ is a reduced homomorphism of $g$. If $g_2|((\alpha) = g_2(\alpha) = 0$ for some $\alpha \in (0 :_{B_1} a)$, then $\alpha \in \text{Ker} g_2 = \text{im} g_1$ and hence there is $\beta \in B_0$ with $\alpha = g_1(\beta)$. It follows that $a\beta \in \text{Ker} g_1 = \text{im} g_0$ and so there is $\gamma \in A$ with $a\beta = g_0(\gamma)$. Since $a$ is an $A$-coregular element, we have $A = aA$ (see [3, Definition 2.4]), and so there is $\gamma' \in A$ with $\gamma = a\gamma'$. It follows that $a(\beta - g_0(\gamma')) = 0$. Hence $\beta - g_0(\gamma') \in (0 :_{B_0} a)$, and $\alpha = g_1(\beta - g_0(\gamma')) = g_1(\beta) \in \text{im} g_1| I$, as desired. \[\square\]

Corollary 2.8. Let $A$ be an Artinian $R$-module and $I = (a_1, ... , a_t)$ an ideal generated by an $A$-coregular sequence. Then $\text{Ext}_A^1(R/I, A) = 0$. 

Proof. From the exact sequence $0 \rightarrow A \rightarrow E \rightarrow K \rightarrow 0$, where $E$ is an injective $R$-module and by using Proposition 2.7, we obtain the result. \[ \square \]

**Question 2.9.** (with hypothesis as Proposition 2.7) Is the following exact sequence for all $n \geq 1$?

$$0 \rightarrow (0 : A I^n) \xrightarrow{g_0} (0 : B_0 I^n) \xrightarrow{g_1} (0 : B_1 I^n) \xrightarrow{g_2} (0 : B_1 I^n),$$

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**References**


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