



## NOTE ON REGULAR AND COREGULAR SEQUENCES

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**ABSTRACT.** Let  $R$  be a commutative Noetherian ring and let  $M$  be a finitely generated  $R$ -module. If  $I$  is an ideal of  $R$  generated by  $M$ -regular sequence, then we study the vanishing of the first Tor functors. Moreover, for Artinian modules and coregular sequences we examine the vanishing of the first Ext functors.

### 1. INTRODUCTION

Throughout this note, we shall assume that  $R$  is a commutative Noetherian ring with non-zero identity and  $I$  an ideal of  $R$ . Suppose  $M$  and  $A$  two non-zero finitely generated and Artinian  $R$ -modules, respectively. An ordered sequence  $x_1, \dots, x_t$  of elements of  $R$  is called an  $M$ -regular sequence [1, Definition 1.1.1]; if  $x_i$  is not a zero-divisor in  $M/(x_1, \dots, x_{i-1})M$  for  $i = 1, \dots, t$ , and  $M \neq (x_1, \dots, x_t)M$ . Also, an ordered sequence  $x_1, \dots, x_t$  of elements of  $R$  is said an  $A$ -coregular sequence [3, Definition 3.1]; if  $x_i$  is an  $(0 :_A (x_1, \dots, x_{i-1}))$ -coregular element for  $i = 1, \dots, t$ , and  $(0 :_A (x_1, \dots, x_t)) \neq 0$ . Note that an element  $x$  of  $R$  is  $A$ -coregular if  $A = xA$ , (see [3, Definition 2.4]).

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It is known that if  $I$  is generated by an  $M$ -regular sequence, then  $\text{Tor}_1^R(R/I, M) = 0$ . In particular, if  $x_1, \dots, x_t$  is an  $R$ -regular sequence,  $I = (x_1, \dots, x_s)$  and  $J = (x_{s+1}, \dots, x_t)$  two ideals of  $R$  then  $\text{Tor}_1^R(R/I, R/J) = 0$  and so  $I \cap J = IJ$ .

Our main aim in this article is to extend Proposition 1.1.4 of [1] for powers an ideal that generated by regular sequence and use it over Tor functors. Additionally, we examine the problem on the self-duality.

## 2. THE RESULTS

The interplay between regular sequences and homological invariants is a major theme, and numerous arguments will be based on the following proposition which is appear in [1].

**Proposition 2.1.** *Let  $R$  be a ring,  $M$  an  $R$ -module, and  $x$  an  $M$ -regular sequence. Then an exact sequence*

$$N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \longrightarrow 0,$$

*induces an exact sequence*

$$\frac{N_2}{xN_2} \longrightarrow \frac{N_1}{xN_1} \longrightarrow \frac{N_0}{xN_0} \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

The following result is a generalization of Proposition 2.1.

**Theorem 2.2.** *Let  $(R, \mathfrak{m})$  be a local ring,  $M$  a finitely generated  $R$ -module, and  $I = (a_1, \dots, a_t)$  an ideal generated by an  $M$ -regular sequence. Then an exact sequence*

$$N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \longrightarrow 0,$$

*of  $R$ -modules induces an exact sequence*

$$\frac{N_2}{I^n N_2} \xrightarrow{\overline{f_2}} \frac{N_1}{I^n N_1} \xrightarrow{\overline{f_1}} \frac{N_0}{I^n N_0} \xrightarrow{\overline{f_0}} \frac{M}{I^n M} \longrightarrow 0,$$

*for all  $n \geq 1$ .*

*Proof.* Since tensor product is a right exact functor, we only need to verify exactness at  $\frac{N_1}{I^n N_1}$ . let  $\overline{\phantom{x}}$  denote residue classes modulo  $I^n N_1$ . Let  $y_1 \in N_1$  such that  $\overline{f_1}(\overline{y}) = 0$ . Then  $f_1(y) \in I^n N_0$ . Hence we can consider a homogeneous polynomial of degree  $n$ ,  $f_1(y) \in N_0[x_1, \dots, x_t]$  such that

$$f_1(y) = \sum_{\alpha_{i_1} + \dots + \alpha_{i_t} = n} \alpha_{i_1 \dots i_t} x_1^{\alpha_{i_1}} \dots x_t^{\alpha_{i_t}} \quad (*)$$

for some  $\alpha_{i_1 \dots i_t} \in N_0$ . Since

$$\sum_{\alpha_{i_1} + \dots + \alpha_{i_t} = n} f_0(\alpha_{i_1 \dots i_t}) x_1^{\alpha_{i_1}} \dots x_t^{\alpha_{i_t}} = f_0 \circ f_1(y) = 0 \in I^{n+1} M, \quad (**)$$

by [1, Theorem 1.1.7] or [2, Theorem 16.2] the coefficients  $f_0(\alpha_{i_1 \dots i_t}) \in IM$ , and so  $f_0(\alpha_{i_1 \dots i_t}) = \sum_{j=1}^t m_{ij} x_j$  for some  $m_{ji} \in M$ . By replacing  $m_{ji}$  in the equation (\*\*) and

repeating this argument, we have  $f_0(\alpha_{i_1 \dots i_t}) \in I^n M$  for all  $n \in \mathbb{N}$ . By Krull intersection Theorem,  $f_0(\alpha_{i_1 \dots i_t}) = 0$  and by our hypothesis  $\alpha_{i_1 \dots i_t} \in \text{im } f_1$ . Hence there exists  $\beta_{i_1 \dots i_t} \in N_1$  such that  $\alpha_{i_1 \dots i_t} = f_1(\beta_{i_1 \dots i_t})$ . Therefore by replacing this value in the equation (\*) and our hypothesis, there exists  $z \in N_2$  such that  $y - \sum_{\alpha_{i_1} + \dots + \alpha_{i_t} = n} \beta_{i_1 \dots i_t} x_1^{\alpha_{i_1}} \dots x_t^{\alpha_{i_t}} = f_2(z)$ . Hence  $\bar{y} = \overline{f_2(z)}$ , as desired.  $\square$

**Corollary 2.3.** *Let  $(R, \mathfrak{m})$  be a local ring,  $M$  a finitely generated  $R$ -module and  $x_1, \dots, x_s, x_{s+1}, \dots, x_t$  an  $M$ -regular sequence. Then the following statements hold:*

- (i) *If  $I = (x_1, \dots, x_s)$ , then  $\text{Tor}_1^R(R/I^n, M) = 0$  for all  $n \geq 1$ .*
- (ii) *If  $J = (x_{s+1}, \dots, x_t)$ , then  $\text{Tor}_1^R(R/J^m, M/I^n M) = 0$  for all  $m, n \geq 1$ .*

*Proof.* (i) From the exact sequence  $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ , where  $F$  is a free  $R$ -module and by using Theorem 2.1, we obtain the result.

(ii) It is known that  $\text{Ass}(M/I^n M) = \text{Ass}(M/IM)$ . It therefore follows

$\text{Ass}(M/I^n M + (x_{s+1}, \dots, x_t)M) = \text{Ass}(M/IM + (x_{s+1}, \dots, x_t)M)$  for all  $n \in \mathbb{N}$  and  $s+1 \leq i \leq t$ . Hence  $x_{s+1}, \dots, x_t$  is an  $M/I^n M$ -regular sequence for all  $n \in \mathbb{N}$  and so by Theorem 2.2, the result follows.  $\square$

**Corollary 2.4.** *Let  $(R, \mathfrak{m})$  be a local ring and  $J \subseteq I$  be two ideals of  $R$  such that  $I/J$  is generated by  $R/J$ -regular sequence. Then  $I^n \cap J = I^{n-1}J$  for all  $n \in \mathbb{N}$ .*

*Proof.* From the exact sequence

$$0 \longrightarrow \mathfrak{m}I + J/\mathfrak{m}I \longrightarrow I/\mathfrak{m}I \longrightarrow I/\mathfrak{m}I + J \longrightarrow 0,$$

and by using Nakayama's Lemma, There exists  $x_1, \dots, x_s, y_1, \dots, y_t$  of  $R$  such that  $I = (x_1, \dots, x_s, y_1, \dots, y_t)$ ,  $J = (x_1, \dots, x_s)$  and  $I/J = (\bar{y}_1, \dots, \bar{y}_t)$ , where  $y_1, \dots, y_t$  is an  $R/J$ -regular sequence. By Corollary 2.3(i)  $\text{Tor}_1^R(R/(y_1, \dots, y_t)^n, R/J) = 0$  for all  $n \in \mathbb{N}$  and so  $J \cap (y_1, \dots, y_t)^n = J(y_1, \dots, y_t)^n$ . Therefore  $J \cap I^n = J \cap (J^n + J^{n-1}(y_1, \dots, y_t) + \dots + J(y_1, \dots, y_t)^{n-1} + (y_1, \dots, y_t)^n) = J^n + J^{n-1}(y_1, \dots, y_t) + \dots + J(y_1, \dots, y_t)^{n-1} + J \cap (y_1, \dots, y_t)^n = JJ^{n-1}$ .  $\square$

**Proposition 2.5.** *Let  $M$  be a finitely generated  $R$ -module and  $I = (a_1, \dots, a_t)$  an ideal generated by an  $M$ -regular sequence and an  $R$ -regular sequence. Then  $\text{Tor}_i^R(\frac{R}{I^n}, M) = 0$  for all  $n \geq 1$  and all  $i \geq 1$ .*

*Proof.* We use induction on  $n$ . If  $n = 1$ , then  $\text{Tor}_i^R(\frac{R}{I}, M) = 0$  for all  $i \geq 1$ , by Theorem 2.2. let  $n > 1$  and suppose that the proposition holds for all integer smaller than  $n$ . By using the exact sequence

$$0 \longrightarrow \frac{I^{n-1}}{I^n} \longrightarrow \frac{R}{I^n} \longrightarrow \frac{R}{I^{n-1}} \longrightarrow 0,$$

and the inductive hypothesis,  $\text{Tor}_i^R(\frac{R}{I^n}, M) \cong \text{Tor}_i^R(\frac{I^{n-1}}{I^n}, M)$  for all  $i \geq 1$ . Hence, by using [1, Theorem 1.1.8] we have the isomorphism  $\frac{I^{n-1}}{I^n} \cong \bigoplus_{j=1}^{(n+t-2)} \frac{R}{I}$ . Therefore, we obtain  $\text{Tor}_i^R(\frac{R}{I^n}, M) = 0$  for all  $n \geq 1$  and all  $i \geq 1$ . This completes the inductive step.  $\square$

**Corollary 2.6.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring,  $M$  a finitely generated maximal Cohen-Macaulay  $R$ -module, and  $I = (a_1, \dots, a_t)$  an ideal generated by an  $R$ -regular sequence. Then  $\text{Tor}_i^R(\frac{R}{I^n}, M) = 0$  for all  $i \geq 1$  and all  $n \geq 1$ .*

*Proof.* Since every  $R$ -regular sequence is an  $M$ -regular sequence, the result follows by Proposition 2.5.  $\square$

The following result is a dual of [1, Proposition 1.1.4].

**Proposition 2.7.** *Let  $A$  be an Artinian  $R$ -module and  $I = (a_1, \dots, a_t)$  an ideal generated by an  $A$ -coregular sequence. Then an exact sequence*

$$0 \longrightarrow A \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2$$

*of  $R$ -modules induces an exact sequence*

$$0 \longrightarrow (0 :_A I) \xrightarrow{g_0|} (0 :_{B_0} I) \xrightarrow{g_1|} (0 :_{B_1} I) \xrightarrow{g_2|} (0 :_{B_2} I).$$

*Proof.* By induction, it is enough to consider the case in which  $I$  consists of a single  $A$ -coregular element  $a = a_1$ . We obtain the induced sequence if we use the functor  $\text{Hom}_R(\frac{R}{I}, -)$ . Since  $\text{Hom}_R(\frac{R}{I}, -)$  is a left exact functor, we only need to verify exactness at  $(0 :_{B_1} a)$ . Let  $g|$  is a reduced homomorphism of  $g$ . If  $g_2|(\alpha) = g_2(\alpha) = 0$  for some  $\alpha \in (0 :_{B_1} a)$ , then  $\alpha \in \text{Ker } g_2 = \text{im } g_1$  and hence there is  $\beta \in B_0$  with  $\alpha = g_1(\beta)$ . It follows that  $a\beta \in \text{Ker } g_1 = \text{im } g_0$  and so there is  $\gamma \in A$  with  $a\beta = g_0(\gamma)$ . Since  $a$  is an  $A$ -coregular element, we have  $A = aA$  (see [3, Definition 2.4]), and so there is  $\gamma' \in A$  with  $\gamma = a\gamma'$ . It follows that  $a(\beta - g_0(\gamma')) = 0$ . Hence  $\beta - g_0(\gamma') \in (0 :_{B_0} a)$ , and  $\alpha = g_1(\beta - g_0(\gamma')) = g_1(\beta) \in \text{im } g_1|$ , as desired.  $\square$

**Corollary 2.8.** *Let  $A$  be an Artinian  $R$ -module and  $I = (a_1, \dots, a_t)$  an ideal generated by an  $A$ -coregular sequence. Then  $\text{Ext}_R^1(\frac{R}{I}, A) = 0$ .*

*Proof.* From the exact sequence  $0 \longrightarrow A \longrightarrow E \longrightarrow K \longrightarrow 0$ , where  $E$  is an injective  $R$ -module and by using Proposition 2.7, we obtain the result.  $\square$

**Question 2.9.** (with hypothesis as Proposition 2.7) Is the following

$$0 \longrightarrow (0 :_A I^n) \xrightarrow{g_0|} (0 :_{B_0} I^n) \xrightarrow{g_1|} (0 :_{B_1} I^n) \xrightarrow{g_2|} (0 :_{B_1} I^n),$$

exact sequence for all  $n \geq 1$ ?

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