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# NOTE ON REGULAR AND COREGULAR SEQUENCES

H. SAREMI*

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Abstract. Let $R$ be a commutative Noetherian ring and let $M$ be a finitely generated $R$-module. If $I$ is an ideal of $R$ generated by $M$-regular sequence, then we study the vanishing of the first Tor functors. Moreover, for Artinian modules and coregular sequences we examine the vanishing of the first Ext functors.

## 1. Introduction

Throughout this note, we shall assume that $R$ is a commutative Noetherian ring with non-zero identity and I an ideal of $R$. Suppose $M$ and $A$ two non-zero finitely generated and Artinian $R$-modules, respectively. An ordered sequence $x_1, ..., x_t$ of elements of $R$ is called an $M$-regular sequence [1, Definition 1.1.1]; if $x_i$ is not a zero-divisor in $M/(x_1, ..., x_{i-1})M$ for $i = 1, ..., t$, and $M \neq (x_1, ..., x_t)M$. Also, an ordered sequence $x_1, ..., x_t$ of elements of $R$ is said an $A$-coregular sequence [3, Definition 3.1]; if $x_i$ is an $(0 :_A (x_1, ..., x_{i-1}))$-coregular element for $i = 1, ..., t$, and $(0 :_A (x_1, ..., x_t)) \neq 0$. Note that an element $x$ of $R$ is $A$-coregular if $A = xA$, (see [3, Definition 2.4]).

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∗Corresponding author

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It is known that if $I$ is generated by an $M$-regular sequence, then $\mathrm{Tor}_1^R(R/I, M) = 0$. In particular, if $x_1, ..., x_t$ is an R-regular sequence, $I = (x_1, ..., x_s)$ and $J = (x_{s+1}, ..., x_t)$ two ideals of $R$ then $\mathrm{Tor}_1^R(R/I, R/J) = 0$ and so $I \cap J = IJ$.

Our main aim in this article is to extend Proposition 1.1.4 of [1] for powers an ideal that generated by regular sequence and use it over Tor functors. Additionally, we examine the problem on the self-duality.

## 2. The results

The interplay between regular sequences and homological invariants is a major theme, and numerous arguments will be based on the following proposition which is appear in [1].

**Proposition 2.1.** *Let $R$ be a ring, $M$ an $R$-module, and $x$ an $M$-regular sequence. Then an exact sequence*

$$N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \longrightarrow 0,$$

*induces an exact sequence*

$$\frac{N_2}{xN_2} \longrightarrow \frac{N_1}{xN_1} \longrightarrow \frac{N_0}{xN_0} \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

The following result is a generalization of Proposition 2.1.

**Theorem 2.2.** *Let $(R, \mathfrak{m})$ be a local ring, $M$ a finitely generated $R$-module, and $I = (a_1, ..., a_t)$ an ideal generated by an $M$-regular sequence. Then an exact sequence*

$$N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N_0 \xrightarrow{f_0} M \longrightarrow 0,$$

*of $R$-modules induces an exact sequence*

$$\frac{N_2}{I^nN_2} \xrightarrow{\overline{f_2}} \frac{N_1}{I^nN_1} \xrightarrow{\overline{f_1}} \frac{N_0}{I^nN_0} \xrightarrow{\overline{f_0}} \frac{M}{I^nM} \longrightarrow 0,$$

*for all $n \geq 1$.*

*Proof.* Since tensor product is a right exact functor, we only need to verify exactness at $\frac{N_1}{I^nN_1}$. let $^-$ denote residue classes modulo $I^nN_1$. Let $y_1 \in N_1$ such that $\overline{f_1}(\overline{y}) = 0$. Then $f_1(y) \in I^nN_0$. Hence we can consider a homogeneous polynomial of degree $n$, $f_1(y) \in N_0[x_1, ..., x_t]$ such that

$$f_1(y) = \sum_{\alpha_{i_1}+...+\alpha_{i_t}=n} \alpha_{i_1...i_t} x_1^{\alpha_{i_1}} ... x_t^{\alpha_{i_t}} \quad (*)$$

for some $\alpha_{i_1...i_t} \in N_0$ . Since

$$\sum_{\alpha_{i_1}+...+\alpha_{i_t}=n} f_0(\alpha_{i_1...i_t}) x_1^{\alpha_{i_1}} ... x_t^{\alpha_{i_t}} = f_0 \circ f_1(y) = 0 \in I^{n+1}M, \quad (**)$$

by [1, Theorem 1.1.7] or [2, Theorem 16.2] the coefficients $f_0(\alpha_{i_1...i_t}) \in IM$, and so $f_0(\alpha_{i_1...i_t}) = \sum_{j=1}^{t} m_{i_j} x_j$ for some $m_{j_i} \in M$. By replacing $m_{j_i}$ in the equation $(**)$ and repeating this argument, we have $f_0(\alpha_{i_1...i_t}) \in I^n M$ for all $n \in \mathbb{N}$. By Krull intersection Theorem, $f_0(\alpha_{i_1...i_t}) = 0$ and by our hypothesis $\alpha_{i_1...i_t} \in \operatorname{im} f_1$. Hence there exists $\beta_{i_1...i_t} \in N_1$ such that $\alpha_{i_1...i_t} = f_1(\beta_{i_1...i_t})$. Therefore by replacing this value in the equation $(*)$ and our hypothesis, there exists $z \in N_2$ such that $y - \sum_{\alpha_{i_1}+...+\alpha_{i_t}=n} \beta_{i_1...i_t} x_1^{\alpha_{i_1}} ... x_t^{\alpha_{i_t}} = f_2(z)$. Hence $\overline{y} = \overline{f_2}(\overline{z})$, as desired. □

**Corollary 2.3.** *Let $(R, \mathfrak{m})$ be a local ring, $M$ a finitely generated $R$-module and $x_1, ..., x_s, x_{s+1}, ..., x_t$ an $M$-regular sequence. Then the following statements hold:*

(i) *If $I = (x_1, ..., x_s)$, then $\operatorname{Tor}_1^R(R/I^n, M) = 0$ for all $n \geq 1$.*
(ii) *If $J = (x_{s+1}, ..., x_t)$, then $\operatorname{Tor}_1^R(R/J^m, M/I^n M) = 0$ for all $m, n \geq 1$.*

*Proof.* $(i)$ From the exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where $F$ is a free $R$-module and by using Theorem 2.1, we obtain the result.
$(ii)$ It is known that $\operatorname{Ass}(M/I^n M) = \operatorname{Ass}(M/IM)$. It therefore follows
$\operatorname{Ass}(M/I^n M + (x_{s+1}, ..., x_i)M) = \operatorname{Ass}(M/IM + (x_{s+1}, ..., x_i)M)$ for all $n \in \mathbb{N}$ and $s+1 \leq i \leq t$. Hence $x_{s+1}, ..., x_t$ is an $M/I^n M$-regular sequence for all $n \in \mathbb{N}$ and so by Theorem 2.2, the result follows. □

**Corollary 2.4.** *Let $(R, \mathfrak{m})$ be a local ring and $J \subseteq I$ be two ideals of $R$ such that $I/J$ is generated by $R/J$-regular sequence. Then $I^n \cap J = I^{n-1} J$ for all $n \in \mathbb{N}$.*

*Proof.* From the exact sequence

$$0 \longrightarrow \mathfrak{m}I + J/\mathfrak{m}I \longrightarrow I/\mathfrak{m}I \longrightarrow I/\mathfrak{m}I + J \longrightarrow 0,$$

and by using Nakayama's Lemma, There exists $x_1, ..., x_s, y_1, ..., y_t$ of $R$ such that $I = (x_1, ..., x_s, y_1, ..., y_t)$, $J = (x_1, ..., x_s)$ and $I/J = (\overline{y_1}, ..., \overline{y_t})$, where $y_1, ..., y_t$ is an $R/J$-regular sequence. By Corollary 2.3(i) $\operatorname{Tor}_1^R(R/(y_1, ..., y_t)^n, R/J) = 0$ for all $n \in \mathbb{N}$ and so $J \cap (y_1, ..., y_t)^n = J(y_1, ..., y_t)^n$. Therefore $J \cap I^n = J \cap (J^n + J^{n-1}(y_1, ..., y_t) + ... + J(y_1, ..., y_t)^{n-1} + (y_1, ..., y_t)^n) = J^n + J^{n-1}(y_1, ..., y_t) + ... + J(y_1, ..., y_t)^{n-1} + J \cap (y_1, ..., y_t)^n = JI^{n-1}$. □

**Proposition 2.5.** *Let $M$ be a finitely generated $R$-module and $I = (a_1, ..., a_t)$ an ideal generated by an $M$-regular sequence and an $R$-regular sequence. Then $\operatorname{Tor}_i^R(\frac{R}{I^n}, M) = 0$ for all $n \geq 1$ and all $i \geq 1$.*

*Proof.* We use induction on $n$. If $n = 1$, then $\mathrm{Tor}_i^R(\frac{R}{I}, M) = 0$ for all $i \geq 1$, by Theorem 2.2. let $n > 1$ and suppose that the proposition holds for all integer smaller than $n$. By using the exact sequence

$$0 \longrightarrow \frac{I^{n-1}}{I^n} \longrightarrow \frac{R}{I^n} \longrightarrow \frac{R}{I^{n-1}} \longrightarrow 0,$$

and the inductive hypothesis, $\mathrm{Tor}_i^R(\frac{R}{I^n}, M) \cong \mathrm{Tor}_i^R(\frac{I^{n-1}}{I^n}, M)$ for all $i \geq 1$. Hence, by using [1, Theorem 1.1.8] we have the isomorphism $\frac{I^{n-1}}{I^n} \cong \oplus_{j=1}^{\binom{n+t-2}{t-1}} \frac{R}{I}$. Therefore, we obtain $\mathrm{Tor}_i^R(\frac{R}{I^n}, M) = 0$ for all $n \geq 1$ and all $i \geq 1$. This completes the inductive step. □

**Corollary 2.6.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ a finitely generated maximal Cohen-Macaulay $R$-module, and $I = (a_1, ..., a_t)$ an ideal generated by an $R$-regular sequence. Then $\mathrm{Tor}_i^R(\frac{R}{I^n}, M) = 0$ for all $i \geq 1$ and all $n \geq 1$.*

*Proof.* Since every $R$-regular sequence is an $M$-regular sequence, the result follows by Proposition 2.5. □

The following result is a dual of [1, Proposition 1.1.4].

**Proposition 2.7.** *Let $A$ be an Artinian $R$-module and $I = (a_1, ..., a_t)$ an ideal generated by an $A$-coregular sequence. Then an exact sequence*

$$0 \longrightarrow A \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2$$

*of $R$-modules induces an exact sequence*

$$0 \longrightarrow (0 :_A I) \xrightarrow{g_0|} (0 :_{B_0} I) \xrightarrow{g_1|} (0 :_{B_1} I) \xrightarrow{g_2|} (0 :_{B_1} I).$$

*Proof.* By induction, it is enough to consider the case in which $I$ consists of a single $A$-coregular element $a = a_1$. We obtain the induced sequence if we use the functor $\mathrm{Hom}_R(\frac{R}{I}, -)$. Since $\mathrm{Hom}_R(\frac{R}{I}, -)$ is a left exact functor, we only need to verify exactness at $(0 :_{B_1} a)$. Let $g|$ is a reduced homomorphism of $g$. If $g_2|(\alpha) = g_2(\alpha) = 0$ for some $\alpha \in (0 :_{B_1} a)$, then $\alpha \in \operatorname{Ker} g_2 = \operatorname{im} g_1$ and hence there is $\beta \in B_0$ with $\alpha = g_1(\beta)$. It follows that $a\beta \in \operatorname{Ker} g_1 = \operatorname{im} g_0$ and so there is $\gamma \in A$ with $a\beta = g_0(\gamma)$. Since $a$ is an $A$-coregular element, we have $A = aA$ (see [3, Definition 2.4]), and so there is $\gamma' \in A$ with $\gamma = a\gamma'$. It follows that $a(\beta - g_0(\gamma')) = 0$. Hence $\beta - g_0(\gamma') \in (0 :_{B_0} a)$, and $\alpha = g_1(\beta - g_0(\gamma')) = g_1(\beta) \in \operatorname{im} g_1|$, as desired. □

**Corollary 2.8.** *Let $A$ be an Artinian $R$-module and $I = (a_1, ..., a_t)$ an ideal generated by an $A$-coregular sequence. Then $\mathrm{Ext}_R^1(\frac{R}{I}, A) = 0$.*

*Proof.* From the exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow K \longrightarrow 0$, where $E$ is an injective $R$-module and by using Proposition 2.7, we obtain the result. □

**Question 2.9.** *(with hypothesis as Proposition 2.7) Is the following*

$$0 \longrightarrow (0:_A I^n) \xrightarrow{g_0|} (0:_{B_0} I^n) \xrightarrow{g_1|} (0:_{B_1} I^n) \xrightarrow{g_2|} (0:_{B_1} I^n),$$

*exact sequence for all $n \geq 1$?*

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## References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, UK, (1998).

[2] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, UK, (1986).

[3] A. Ooishi, *Matlis duality and width of a module*, Hiroshima Math. J., **6** (1976), 573-587.

**Hero Saremi**

Department of Mathematics, Sanandaj Branch

Islamic Azad University

Sanandaj, Iran.

hero.saremi@gmail.com