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ON THE EIGENVALUES OF NON-COMMUTING GRAPHS

MODJTABA GHORBANI, ZAHRA GHARAVI-ALKHANSARI AND ALI ZAEEM-BASHI

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Abstract. The non-commuting graph $\Gamma(G)$ of a non-abelian group G with the center $Z(G)$ is a graph with the vertex set $V(\Gamma(G)) = G \setminus Z(G)$ and two distinct vertices x and y are adjacent in $\Gamma(G)$ if and only if $xy \neq yx$. The aim of this paper is to compute the spectra of some well-known NC-graphs.

1. Introduction

All graphs considered in this paper are simple namely undirected graph without parallel edges. Also, all graphs and groups are finite. Let G be a non-abelian group with the center $Z(G)$. The non-commuting graph ($NC-$graph) $\Gamma(G)$ is a graph with the vertex set $G \setminus Z(G)$ and two distinct vertices $x, y \in G \setminus Z(G)$ are adjacent whenever $xy \neq yx$. The concept of $NC-$graphs was first considered by Paul Erdős in 1975 to answer a question on the size of the cliques of a graph, see [21]. For background materials about $NC-$graphs, we encourage the reader to see references [1, 12, 19, 20].

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∗Corresponding author: mghorbani@sru.ac.ir

In the next section, we give necessary definitions and some preliminary results and the third section contains the main results on the spectra of NC-graphs.

2. Definitions and Preliminaries

Our notation is standard and mainly taken from standard books such as [7, 8, 11]. For a group G, $Cent(G) = \{C_G(x) | x \in G\}$, where $C_G(x)$ is the centralizer of the element x in G, namely $C_G(x) = \{y \in G | xy = yx\}$, see [2, 4, 6].

Example 2.1. Consider the symmetric group $\mathbb{S}_3$ by the following presentation

$$\mathbb{S}_3 = \langle a, b : \ a^2 = 1, b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$

This group is the smallest non-abelian group of order 6. The center of this group is trivial and so $\mathbb{S}_3 \setminus Z(\mathbb{S}_3) = \{a, b, b^2, ab, ab^2\}$. The element b commutes with b^2 and thus $\Gamma(\mathbb{S}_3) \cong K_5 \setminus e$, where $K_n \setminus e$ denotes the graph obtained from the complete graph K_n by deleting an edge.

An independent set of a graph Γ is a subset $S \subseteq V(\Gamma)$ if no two vertices of which are adjacent. The size of the largest independent set is called the independence number. A k-partite graph is a graph whose vertices can be partitioned into k different independent sets. When $k = 2$ or 3, the related graph is denoted by bipartite or tripartite graph, respectively.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of sizes m by p and q by n, respectively. The *tensor product* (or *Kronecker product*) of A and B is the mq by pn matrix $A \otimes B$ obtained from A by replacing each entry a_{ij} of A with the q by n matrix

$$a_{ij}B \ (1 \leq i \leq m, 1 \leq j \leq p).$$

The *lexicographic product* or *composition* graph $\Gamma_1 \ \text{o} \ \Gamma_2$ of two graphs Γ_1 and Γ_2, is a graph with the vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and any two vertices (u, v) and (x, y) are adjacent in $\Gamma_1 \ \text{o} \ \Gamma_2$ if and only if either u is adjacent with x in Γ_1 or $u = x$ and v is adjacent with y in Γ_2. If the adjacency matrices of two graphs Γ_1 and Γ_2 are $A_{m\times m}$ and $B_{n\times n}$ respectively, then the lexicographic product of $\Gamma_1 \ \text{o} \ \Gamma_2$ has adjacency matrix

$$A \otimes J_m + I_n \otimes B.$$

For given graphs Γ_1 and Γ_2 their *Cartesian product* $\Gamma_1 \Box \Gamma_2$ is defined as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if either ($[u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2)]$) or ($[u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1)]$). Let A and B be square matrices of orders m and n, respectively. The adjacency matrix of Cartesian product $\Gamma_1 \Box \Gamma_2$ can be written as $A \otimes I_m + I_n \otimes B$, see [8].

The *direct product* $\Gamma_1 \boxtimes \Gamma_2$ of two graphs Γ_1 and Γ_2 is defined as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if

$u_1v_1 \in E(\Gamma_1)$ and $u_2v_2 \in E(\Gamma_2)$. The adjacency matrix of $\Gamma_1 \boxtimes \Gamma_2$ is the tensor product $A \otimes B$ of the adjacency matrices of Γ_1 and Γ_2.

Example 2.2. Consider the group U_{6n} with the following presentation

$$U_{6n} = \langle a, b : \ a^{2n} = 1, b^3 = 1, a^{-1}ba = b^{-1}\rangle.$$

The elements of this group are

$$\{1, a, \cdots, a^{2n-1}, b, ba, \cdots, ba^{2n-1}, b^2, b^2a, \cdots, b^2a^{2n-1}\}.$$

One can see that $Z(U_{6n}) = \langle a^2\rangle$ and so $|Z(U_{6n})| = n$. This implies that

$$|V(\Gamma(U_{6n}))| = |U_{6n}| - |Z(U_{6n})| = 5n.$$

Let i, j be odd numbers, then

$$(a^ib)(a^jb) = (a^ib)a(a^{j-1}b) = a^i(ba)a^{j-1}b = a^{i+j} = (a^jb)(a^ib).$$

Hence, $\{ab, a^3b, \cdots, a^{2n-1}b\}$ is an independent set. Similarly, we can prove that if i, j are odd numbers, then $(a^ib^2)(a^jb^2) = (a^jb^2)(a^ib^2)$ and so the set $\{ab^2, \cdots, a^{2n-1}b^2\}$ is an independent set. Now we can show that the following sets are independent

$$\{a, a^3, \cdots, a^{2n-1}\}, \{ab, a^3b, \cdots, a^{2n-1}b\}, \{ab^2, a^3b^2, \cdots, a^{2n-1}b^2\},$$
$$\{b, b^2, a^2b, a^2b^2 \cdots, a^{2n-2}b, a^{2n-2}b^2\}.$$

This implies that $\Gamma(U_{6n})$ is a 4-partite graph with the following adjacency matrix

$$\begin{pmatrix} 0_n & J_n & J_n & J_{n\times 2n} \\ J_n & 0_n & J_n & J_{n\times 2n} \\ J_n & J_n & 0_n & J_{n\times 2n} \\ J_{2n\times n} & J_{2n\times n} & J_{2n\times n} & 0_{2n} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \otimes J_n = B \otimes J_n,$$

where J_n is the square matrix with all entries one.

Example 2.3. Consider now the NC-graph of group T_{4n} with the following presentation

$$T_{4n} = \langle a, b : \ a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle.$$

The elements of the this group are

$$\{1, a, \cdots, a^{2n-1}, b, ba, \cdots, ba^{2n-1}\}.$$

One can prove that $Z(T_{4n}) = \langle b^2\rangle$ and so $|Z(T_{4n})| = 2$. This implies that

$$|V(\Gamma(T_{4n}))| = |T_{4n}| - |Z(T_{4n})| = 4n - 2.$$

It is not difficult to see that, $\Gamma(T_{4n})$ is $(n+1)$-partite graph. On the other hand, Γ has $2n-2$ vertices of degree $2n$ and $2n$ vertices of degree $4n-4$. This implies that the adjacency matrix of $\Gamma(T_{4n})$ is

$$\begin{pmatrix} 0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\ \vdots & & & & & \\ 0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\ J_2 & \cdots & J_2 & 0_2 & \cdots & J_2 \\ \vdots & & & & & \\ J_2 & \cdots & J_2 & J_2 & \cdots & 0_2 \end{pmatrix} = \begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix} \otimes J_2.$$

We recall that a finite group is called a p-group if and only if its order is a power of p, where p is a prime integer. In [13], it is proved that there is no regular $NC-$graph of valency p^n, where p is an odd prime number and n is a positive integer. In general, we have the following result.

Theorem 2.4. [13] *Let G be a finite non-abelian group such that $\Gamma(G)$ is k-regular. Then k is an even number greater than or equal with 4.*

Theorem 2.5. [13] *Let G be a finite non-abelian group such that $\Gamma(G)$ is 2^s-regular, where $s \in \mathbb{N} \setminus \{1\}$. Then G is a 2-group.*

Proposition 2.6. [1] *Let G be a finite non-abelian group such that $\Gamma(G)$ is a regular graph. Then G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group, P is a p-group (p is a prime) and furthermore $\Gamma(P)$ is a regular graph.*

Theorem 2.7. [14] *Let G be a non-abelian group and p be a prime number. If $[G : Z(G)] = p^2$, then $\Gamma(G)$ is a regular graph.*

Theorem 2.8. [6] *Let G be a finite non-abelian group. Then $|Cent(G)| = 4$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 2.9. [6] *Let G be a finite non-abelian group and p be a prime number. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $|Cent(G)| = p + 2$.*

Remark 2.10. Let $G \cong P \times \mathbb{Z}_q$ where p, q are prime numbers and P be a p-group. Hence, we have $G/Z(G) \cong P/Z(P)$. Thus, $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Proposition 2.11. [13] *Let p be a prime number and P be a non-abelian p-group. Then $\Gamma(P)$ is k-regular if and only if $\Gamma(P \times \mathbb{Z}_q)$ is kq-regular, where q is a prime number.*

In the following by $\overline{K}_n$ we mean the complement of the complete graph K_n.

Corollary 2.12. [14] *Let p be a prime number and P be a non-abelian p-group. If $G = P \times A$, where A is an abelian group, then the graph $\Gamma(G)$ is lexicographic product of $\Gamma(P)$ around $\overline{K}_{|A|}$ i.e. $\Gamma(G) \cong \Gamma(P)\ o\ \overline{K}_{|A|}$.*

Theorem 2.13. [14] *Let G be a finite non-abelian group and p be a prime number. Then $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $\Gamma(G)$ is a regular complete $(p+1)$-partite graph.*

3. Main Results

Let Γ be a graph with adjacency matrix A, the characteristic polynomial of Γ is defined as $\chi_\Gamma(\lambda) = det(\lambda I - A)$, where I is the identity matrix. The roots of this polynomial are called the eigenvalues of Γ and form the spectrum of this graph, see [4, 10, 11, 15, 16, 17]. It is a well-known fact that if A is a real symmetric matrix, then all eigenvalues of A are real. The graph Γ is said to be integral if all its eigenvalues are integers, see [3, 5, 9, 10, 18].

Proposition 3.1. [22] *A graph has exactly one positive eigenvalue if and only if the non-isolated vertices form a complete multipartite graph.*

Lemma 3.2. [11] *Let M be the following block matrix:*

$$M = \begin{pmatrix} 0_{m\times m} & B_{m\times n} \\ B^T_{n\times m} & A_{n\times n} \end{pmatrix}.$$

Then

$$\chi_M(\lambda) = |\lambda I - M| = \lambda^{m-n}|\lambda^2 I_n - \lambda A - B^T B|.$$

Theorem 3.3. [8] *Let A and B be square matrices of orders m and n, respectively. If $\lambda_1, \cdots, \lambda_m$ are eigenvalues of A and $\mu_1, \cdots, \mu_n$ are eigenvalues of B, then for $1 \le i \le m, 1 \le j \le n$, the eigenvalues of $A \otimes B$ are $\lambda_i\mu_j$ and the eigenvalues of $A \otimes I_m + I_n \otimes B$ are $\lambda_i + \mu_j$.*

The aim of this section is to study the spectral properties of $NC-$graphs. In [1] it is proved that the diameter of an $NC-$graph is two. On the other hand, in [10] it is proved that if Γ is an integral $k-$regular graph on n vertices with diameter d, then

$$n \le \frac{k(k-1)^d - 2}{k-2}.$$

By using these results, in [13] the authors has proposed a necessary condition for $\Gamma(G)$ to be an integral k-regular graph. Here, we give a sufficient condition for $\Gamma(G)$ to be integral.

Theorem 3.4. *Let G be a finite non-abelian group and p be a prime number. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\Gamma(G)$ is an integral graph.*

Proof. By Theorem 2.13, $\Gamma(G)$ is a regular complete $(p+1)$-partite graph and so it is a strongly regular graph with parameters (k, λ, μ). Hence, the eigenvalues of $\Gamma(G)$ are as follows:

$$\left\{ \left[\frac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}\right]^{m_1}, \left[\frac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}\right]^{m_2}, [k]^1 \right\}.$$

If n is the number of vertices of the graph, then the number of vertices of each part of this graph is $n/(p+1)$. Hence, we have

$$k = \frac{pn}{p+1}, \ \lambda = \frac{(p-1)n}{p+1}, \ \mu = \frac{pn}{p+1}.$$

Therefore the spectrum of $\Gamma(G)$ is

$$Spec(\Gamma(G)) = \left\{ \left[\frac{-n}{p+1}\right]^{m_1}, [0]^{m_2}, \left[\frac{pn}{p+1}\right]^1 \right\}.$$

On the other hand, $m_1+m_2+1 = n$ and $\frac{pn}{p+1}+m_1\frac{-n}{p+1} = 0$. Hence $m_1 = p$ and $m_2 = n-1-p$. Since $p+1$ divides n, the eigenvalues of this graph are integral.□

Corollary 3.5. *Let G be a finite non-abelian group and p be a prime number. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\Gamma(G)$ has only one positive eigenvalue. In particular, $p(p-1)|Z(G)|$ is the only positive eigenvalue of the regular graph $\Gamma(G)$.*

Proof. According to Theorem 2.13, $\Gamma(G)$ is a complete $(p+1)$-partite graph. Thus, by using Proposition 3.1, $\Gamma(G)$ has only one positive eigenvalue. By Theorem 3.4 the positive eigenvalue of $\Gamma(G)$ is

$$\frac{p(|G|-|Z(G)|)}{p+1} = \frac{p(p^2-1)|Z(G)|}{p+1} = p(p-1)|Z(G)|.$$

□

Theorem 3.6. *Let p be a prime number and P be a p-group. If $G = P \times A$ where A is an abelian group, then the spectrum of $\Gamma(G)$ is*

$$\left\{ [0]^{(a-1)|V(\Gamma(P))|}, [a\lambda_1]^{m_1}, \cdots, [a\lambda_s]^{m_s} \right\},$$

where $\{[\lambda_1]^{m_1}, \cdots, [\lambda_s]^{m_s}\}$ is the spectrum of $\Gamma(P)$ and $|A| = a$.

Proof. By Corollary 2.12, we have $\Gamma(G) \cong \Gamma(P)\ o\ \overline{K}_{|A|}$. Let $|A| = a$ and B be the adjacency matrix of $\Gamma(P)$. Since the adjacency matrix of $\overline{K}_{|A|}$ is $(0)_{a\times a}$, the adjacency matrix of $\Gamma(G)$ is $B \otimes J_a$. Since the characteristic polynomial of J_a is $\chi_{J_a}(\lambda) = \lambda^{a-1}(\lambda - a)$, by using Theorem 3.3, the spectrum of $\Gamma(G)$ is

$$Spec(\Gamma(G)) = \left\{ [0]^{(a-1)|V(\Gamma(P))|}, [a\lambda_1]^{m_1}, \cdots, [a\lambda_s]^{m_s} \right\}.$$

□

Lemma 3.7. *Consider the block matrix*

$$A = \begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix}.$$

The characteristic polynomial of A is

$$\chi_A(\lambda) = \lambda^{n-2}(\lambda+1)^{n-1}(\lambda^2+(1-n)\lambda - n(n-1)). \tag{1}$$

Proof. By using Lemma 3.2, we have

$$\chi_A(\lambda) = \lambda^{m-n}|\lambda^2 I_n - \lambda A - BB^T|,$$

where $m = n-1$ and $B = J_{n\times(n-1)}$. Hence,

$$\chi_A(\lambda) = \lambda^{-1}|(\lambda^2+\lambda)I_n - (\lambda+n-1)J_n| = \lambda^{-1}(\lambda+n-1)^n \chi_{\frac{\lambda^2+\lambda}{\lambda+n-1}}(J_n).$$

Since $\chi_{J_n}(\lambda) = \lambda^{n-1}(\lambda - n)$, the proof is complete.□

Theorem 3.8. *The spectrum of $\Gamma(U_{6n})$ is*

$$Spec(\Gamma(U_{6n})) = \left\{[-n]^2, \left[n \pm n\sqrt{7}\right]^1, [0]^{5n-4}\right\}.$$

Proof. In Example 2.2, it is shown that $\Gamma(U_{6n})$ is a 4-partite graph with the adjacency matrix $B \otimes J_n$. The eigenvalues of J_n and B are $\{[0]^{n-1}, [n]^1\}$ and $\{[0]^1, [1+\sqrt{7}]^1, [1-\sqrt{7}]^1, [-1]^2\}$, respectively. Now Theorem 3.3 yeilds the proof.□

Theorem 3.9. *The spectrum of graph $\Gamma(T_{4n})$ is as follows:*

$$Spec(\Gamma(T_{4n})) = \left\{[-2]^{n-1}, [0]^{3n-3}, \left[(n-1) \pm \sqrt{(5n-1)(n-1)}\right]^1\right\}.$$

Proof. In Example 2.3, it is shown that $\Gamma(T_{4n})$ is a $(n+1)$-partite graph with the following adjacency matrix

$$\begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix} \otimes J_2.$$

The spectrum of the left hand matrix can be computed directly from Lemma 3.7 as follows:

$$\left\{[-1]^{n-1}, [0]^{n-2}, \left[\frac{n-1}{2} \pm \frac{\sqrt{(5n-1)(n-1)}}{2}\right]^1\right\}.$$

Thus, by using Theorem 3.3, the proof is complete.□

Theorem 3.10. *The spectrum of $NC-$graph D_{2n} is as follows:*

if n is odd:

$$Spec(\Gamma(D_{2n})) = \left\{ [-1]^{n-1}, [0]^{n-2}, \left[\frac{n-1}{2} \pm \frac{\sqrt{(5n-1)(n-1)}}{2} \right]^1 \right\}.$$

if n is even:

$$Spec(\Gamma(D_{2n})) = \left\{ [-2]^{\frac{n}{2}-1}, [0]^{\frac{3n}{2}-3}, \left[(\frac{n}{2}-1) \pm \sqrt{(\frac{5n}{2}-1)(\frac{n}{2}-1)} \right]^1 \right\}.$$

Proof. In finding the spectrum of $\Gamma(D_{2n})$, it is convenient to consider two separately cases:
Case 1. n is odd, the adjacency matrix of Γ has the following form:

$$\begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix}.$$

By using Lemma 3.7, the proof is complete.
Case 2. $n = 2m$ is even, in this case $\Gamma(D_{2n}) \cong \Gamma(T_{4m})$ and according to Theorem 3.9, the proof is complete.□

Here, we determine the spectrum of $NC-$graph of group V_{8n}(n is odd) with the following presentation:

$$V_{8n} = \langle a, b : \ a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1} \rangle.$$

Theorem 3.11. *The spectrum of $\Gamma(V_{8n})$ is given by*

$$\left\{ [-2]^{2n-1}, [0]^{6n-3}, \left[2n-1 \pm \sqrt{20n^2-12n+1} \right]^1 \right\}.$$

Proof. One can prove that $Z(V_{8n}) = \langle b^2 \rangle$ and so $|Z(V_{8n})| = 2$. This implies that

$$|V(\Gamma(V_{8n}))| = |V_{8n}| - |Z(V_{8n})| = 8n - 2.$$

Similar to the proof of Theorem 3.9 and Theorem 3.10, we can show that $\Gamma(V_{8n})$ is a $(2n+1)$-partite graph with the following partitions:

$$\begin{aligned} V_1 &= \{a, \cdots, a^{2n-1}, ab^2, \cdots, a^{2n-1}b^2\}, \\ V_2 &= \{b, b^3\}, \\ V_3 &= \{ab, ab^3\}, \\ &\vdots \\ V_{2n+1} &= \{a^{2n-1}b, a^{2n-1}b^3\}. \end{aligned}$$

In other words, the vertices of V_1 have degree $4n$ and the other vertices have degree $8n-4$. This implies that its adjacency matrix is $A(\Gamma(V_{8n})) = C \otimes J_2$, where

$$C = \begin{pmatrix} 0_{2n-1} & J_{(2n-1)\times 2n} \\ J_{2n\times(2n-1)} & (J-I)_{2n} \end{pmatrix}.$$

By using Lemma 3.7, we have

$$\chi_C(\lambda) = \lambda^{2n-2}(\lambda+1)^{2n-1}(\lambda^2 + (1-2n)\lambda - 4n^2 + 2n).$$

By computing the roots of above polynomial, the spectrum of C can be computed as follows:

$$\left\{[-1]^{2n-1}, [0]^{2n-2}, \left[(2n-1 \pm \sqrt{20n^2-12n+1})/2\right]^1\right\}.$$

Now, apply Theorem 3.3 to complete the proof. □

In continuing, we determine the spectrum of $NC-$graph of group SD_{8n} with the following presentation:

$$SD_{8n} = \langle a, b : \ a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle.$$

Theorem 3.12. *The spectrum of* $\Gamma(SD_{8n})$ *is as follows:*

if n *is even:*

$$Spec(\Gamma(SD_{8n})) = \left\{[-2]^{2n-1}, [0]^{6n-3}, \left[2n-1 \pm \sqrt{20n^2-12n+1}\right]^1\right\}.$$

if n *is odd:*

$$Spec(\Gamma(SD_{8n})) = \left\{[-4]^{n-1}, [0]^{7n-5}, \left[2(n-1) \pm 2\sqrt{(5n-1)(n-1)}\right]^1\right\}.$$

Proof. One can prove that if n is even, then $Z(SD_{8n}) = \langle a^{2n}\rangle$ and so $|Z(SD_{8n})| = 2$ and if n is odd, then $Z(SD_{8n}) = \langle a^n\rangle$. Thus, $|Z(SD_{8n})| = 4$. This implies that if n is even, then

$$|V(\Gamma(SD_{8n}))| = |SD_{8n}| - |Z(SD_{8n})| = 8n-2,$$

and if n is odd, then

$$|V(\Gamma(SD_{8n}))| = |SD_{8n}| - |Z(SD_{8n})| = 8n-4.$$

We can show that if n is even then $\Gamma(SD_{8n})$ is a $(2n+1)$-partite graph with partitions

$$\begin{aligned} V_1 &= \{a, a^2, \cdots, a^{2n-1}, a^{2n+1}, \cdots, a^{4n-1}\}, \\ V_2 &= \{b, a^{2n}b\}, \\ V_3 &= \{ab, a^{2n+1}b\}, \\ &\vdots \\ V_{2n+1} &= \{a^{2n-1}b, a^{4n-1}b\} \end{aligned}$$

and if n is odd then $\Gamma(SD_{8n})$ is a $(n+1)$-partite graph with partitions

$$\begin{aligned} V_1 &= \{a, a^2, \cdots, a^{4n-1}\} \setminus \{a^n, a^{2n}, a^{3n}\}, \\ V_2 &= \{b, a^n b, a^{2n} b, a^{3n} b\}, \\ V_3 &= \{ab, a^{n+1} b, a^{2n+1} b, a^{3n+1} b\}, \\ &\vdots \\ V_{n+1} &= \{a^{n-1} b, a^{2n-1} b, a^{3n-1} b, a^{4n-1} b\}. \end{aligned}$$

In other words, if n is even, then the vertices of V_1 have degree $4n$ and the other vertices have degree $8n-4$. This implies that its adjacency matrix is equal with $A(\Gamma(V_{8n}))$ and thus $\Gamma(SD_{8n})$ and $\Gamma(V_{8n})$, where n is even, are co-spectral. If n is odd, the vertices of V_1 have degree $4n$ and the other vertices have degree $8n-8$. This implies that its adjacency matrix is $A(\Gamma(SD_{8n})) = C \otimes J_4$, where

$$C = \begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times(n-1)} & (J-I)_n \end{pmatrix}.$$

The spectrum of C can be directly computed by Lemma 3.7 as follows:

$$\left\{ [-1]^{n-1}, [0]^{n-2}, \left[\frac{n-1}{2} \pm \frac{\sqrt{(5n-1)(n-1)}}{2}\right]^1 \right\}.$$

Thus, by using Theorem 3.3, the proof is complete. □

Finally, we determine the spectrum of $NC-$graph of Frobenius group $F_{p,q}$ in which p is prime and $q|p-1$. This group is a non-abelian group of order pq with the following presentation:

$$F_{p,q} = \langle a, b : \ a^p = b^q = 1, b^{-1}ab = a^u \rangle$$

where u is an element of order q in $\mathbb{Z}_p^*$.

Theorem 3.13. *Let $\alpha = (q-1)(p-1)$. The spectrum of $\Gamma(F_{p,q})$ is given by*

$$\left\{ [-(q-1)]^{p-1}, [0]^{pq-p-2}, \left[\frac{\alpha \pm \sqrt{\alpha^2 - 4p\alpha}}{2}\right]^1 \right\}.$$

Proof. It is not difficult to see that $Z(F_{p,q}) = 1$ and therefore $|Z(F_{p,q})| = 1$. The elements of this group are

$$\{1, a, a^2, \cdots, a^{p-1}\} \cup \{a^m b^n; 0 \le m \le p-1, 1 \le n \le q-1\}.$$

Now we compute the centralizer of $a^m b^n$. First notice that

$$[G : C_G(a^m b^n)] = |(a^m b^n)^G| = |(b^n)^G| = p.$$

This implies that $\frac{|G|}{|C_G(a^m b^n|}=p$ and so $|C_G(a^m b^n)|=q$ which yields $\langle a^m b^n\rangle \subseteq C_G(a^m b^n)$. On the other hand, $o(a^m b^n)=q$ and therefore $\langle a^m b^n\rangle = C_G(a^m b^n)$. So the graph $\Gamma(F_{p,q})$ is a multi-partite graph, where one part is of order $p-1$ with the elements $\{1,a,a^2,\cdots,a^{p-1}\}$ and the other parts are of order $q-1$. Clearly, the number of parts of order $q-1$ is

$$\frac{pq-(p-1)-1}{q-1}=\frac{pq-p}{q-1}=p.$$

This implies that

$$A(\Gamma(F_{p,q}))=\begin{pmatrix} 0_{p-1} & J_{(p-1)\times(q-1)} & J_{(p-1)\times(q-1)} & \cdots & J_{(p-1)\times(q-1)} \\ J_{(q-1)\times(p-1)} & 0_{(q-1)} & J_{(q-1)} & \cdots & J_{(q-1)} \\ \vdots & & & & \\ J_{(q-1)\times(p-1)} & J_{(q-1)} & \cdots & \cdots & 0_{(q-1)} \end{pmatrix}.$$

This yields that

$$\begin{aligned} det(xI-A(\Gamma(F_{p,q}))) &= \left|\begin{array}{c|c} xI_{(p-1)} & J_{1\times p}\otimes(-J_{(p-1)\times(q-1)}) \\ \hline J_{p\times 1}\otimes(-J_{(q-1)\times(p-1)}) & I_p\otimes xI_{q-1}+(J-I)_p\otimes(-J_{(q-1)}) \end{array}\right| \\ &= x^{pq-p-2}(x+(q-1))^{p-1}(x-x_1)(x-x_2). \end{aligned}$$

where

$$x_1=\frac{\alpha+\sqrt{\alpha^2-4p\alpha}}{2} \; and \; x_2=\frac{\alpha-\sqrt{\alpha^2-4p\alpha}}{2}.$$

This completes the proof. □

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Modjtaba Ghorbani

Department of Mathematics, Faculty of Science, Shahid Rajaee

Teacher Training University, Tehran, 16785-136, I. R. Iran

mghorbani@sru.ac.ir

Zahra Gharavi

Department of Mathematics, Faculty of Science, Shahid Rajaee

Teacher Training University, Tehran, 16785-136, I. R. Iran

z.gh.alk5@gmail.com

Ali Zaeem-Bashi

Department of Mathematics, Faculty of Science, Shahid Rajaee

Teacher Training University, Tehran, 16785-136, I. R. Iran

zaeembashiali@yahoo.com