ON THE EIGENVALUES OF NON-COMMUTING GRAPHS

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Abstract. The non-commuting graph \( \Gamma(G) \) of a non-abelian group \( G \) with the center \( Z(G) \) is a graph with the vertex set \( V(\Gamma(G)) = G \setminus Z(G) \) and two distinct vertices \( x \) and \( y \) are adjacent in \( \Gamma(G) \) if and only if \( xy \neq yx \). The aim of this paper is to compute the spectra of some well-known NC-graphs.

1. Introduction

All graphs considered in this paper are simple namely undirected graph without parallel edges. Also, all graphs and groups are finite. Let \( G \) be a non-abelian group with the center \( Z(G) \). The non-commuting graph (NC-graph) \( \Gamma(G) \) is a graph with the vertex set \( G \setminus Z(G) \) and two distinct vertices \( x, y \in G \setminus Z(G) \) are adjacent whenever \( xy \neq yx \). The concept of NC-graphs was first considered by Paul Erdős in 1975 to answer a question on the size of the cliques of a graph, see [21]. For background materials about NC-graphs, we encourage the reader to see references [1, 12, 13, 21].

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In the next section, we give necessary definitions and some preliminary results and the third section contains the main results on the spectra of NC-graphs.

2. Definitions and Preliminaries

Our notation is standard and mainly taken from standard books such as [I, S, III]. For a group \( G \), \( \text{Cent}(G) = \{ C_G(x)| x \in G \} \), where \( C_G(x) \) is the centralizer of the element \( x \) in \( G \), namely \( C_G(x) = \{ y \in G | xy = yx \} \), see [A, I, II].

**Example 2.1.** Consider the symmetric group \( S_3 \) by the following presentation

\[
S_3 = \langle a, b : a^2 = 1, b^3 = 1, a^{-1}ba = b^{-1} \rangle.
\]

This group is the smallest non-abelian group of order 6. The center of this group is trivial and so \( S_3 \setminus Z(S_3) = \{ a, b, b^2, ab, ab^2 \} \). The element \( b \) commutes with \( b^2 \) and thus \( \Gamma(S_3) \cong K_5 \setminus e \), where \( K_n \setminus e \) denotes the graph obtained from the complete graph \( K_n \) by deleting an edge.

An independent set of a graph \( \Gamma \) is a subset \( S \subseteq V(\Gamma) \) if no two vertices of which are adjacent. The size of the largest independent set is called the independence number. A \( k \)-partite graph is a graph whose vertices can be partitioned into \( k \) different independent sets. When \( k = 2 \) or \( 3 \), the related graph is denoted by bipartite or tripartite graph, respectively.

Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be matrices of sizes \( m \) by \( p \) and \( q \) by \( n \), respectively. The tensor product (or Kronecker product) of \( A \) and \( B \) is the \( mq \) by \( pn \) matrix \( A \otimes B \) obtained from \( A \) by replacing each entry \( a_{ij} \) of \( A \) with the \( q \) by \( n \) matrix

\[
a_{ij}B \ (1 \leq i \leq m, 1 \leq j \leq p).
\]

The lexicographic product or composition graph \( \Gamma_1 \circ \Gamma_2 \) of two graphs \( \Gamma_1 \) and \( \Gamma_2 \), is a graph with the vertex set \( V(\Gamma_1) \times V(\Gamma_2) \) and any two vertices \( (u, v) \) and \( (x, y) \) are adjacent in \( \Gamma_1 \circ \Gamma_2 \) if and only if either \( u \) is adjacent with \( x \) in \( \Gamma_1 \) or \( u = x \) and \( v \) is adjacent with \( y \) in \( \Gamma_2 \). If the adjacency matrices of two graphs \( \Gamma_1 \) and \( \Gamma_2 \) are \( A_{m \times m} \) and \( B_{n \times n} \) respectively, then the lexicographic product of \( \Gamma_1 \circ \Gamma_2 \) has adjacency matrix

\[
A \otimes J_m + I_n \otimes B.
\]

For given graphs \( \Gamma_1 \) and \( \Gamma_2 \) their Cartesian product \( \Gamma_1 \square \Gamma_2 \) is defined as the graph on the vertex set \( V(\Gamma_1) \times V(\Gamma_2) \), where two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent if and only if either \( ([u_1 = v_1 \text{ and } u_2 v_2 \in E(\Gamma_2)]) \) or \( ([u_2 = v_2 \text{ and } u_1 v_1 \in E(\Gamma_1)]) \). Let \( A \) and \( B \) be square matrices of orders \( m \) and \( n \), respectively. The adjacency matrix of Cartesian product \( \Gamma_1 \square \Gamma_2 \) can be written as \( A \otimes I_m + I_n \otimes B \), see [S].

The direct product \( \Gamma_1 \boxtimes \Gamma_2 \) of two graphs \( \Gamma_1 \) and \( \Gamma_2 \) is defined as the graph on the vertex set \( V(\Gamma_1) \times V(\Gamma_2) \) and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent if and only if
$u_1v_1 \in E(\Gamma_1)$ and $u_2v_2 \in E(\Gamma_2)$. The adjacency matrix of $\Gamma_1 \otimes \Gamma_2$ is the tensor product $A \otimes B$ of the adjacency matrices of $\Gamma_1$ and $\Gamma_2$.

**Example 2.2.** Consider the group $U_{6n}$ with the following presentation

$$U_{6n} = \langle a, b : a^{2n} = 1, b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$ 

The elements of this group are

$$\{1, a, \cdots, a^{2n-1} , b, ba, \cdots, ba^{2n-1}, b^2, b^2a, \cdots, b^2a^{2n-1}\}.$$ 

One can see that $Z(U_{6n}) = \langle a^2 \rangle$ and so $|Z(U_{6n})| = n$. This implies that

$$|V(\Gamma(U_{6n}))| = |U_{6n}| - |Z(U_{6n})| = 5n.$$ 

Let $i, j$ be odd numbers, then

$$(a^i b)(a^j b) = (a^i b)a(a^{j-1}b) = a^i(ba)a^{j-1}b = a^{i+j} = (a^i b)(a^j b).$$ 

Hence, $\{ab, a^3b, \cdots, a^{2n-1}b\}$ is an independent set. Similarly, we can prove that if $i, j$ are odd numbers, then $(a^i b^2)(a^j b^2) = (a^j b^2)(a^i b^2)$ and so the set $\{ab^2, \cdots, a^{2n-1}b^2\}$ is an independent set. Now we can show that the following sets are independent

$$\{a, a^3, \cdots, a^{2n-1}\}, \{ab, a^3b, \cdots, a^{2n-1}b\}, \{ab^2, a^3b^2, \cdots, a^{2n-1}b^2\},$$

$$\{b, b^2, a^2b, a^2b^2 \cdots, a^{2n-2}b, a^{2n-2}b^2\}.$$ 

This implies that $\Gamma(U_{6n})$ is a 4-partite graph with the following adjacency matrix

$$\begin{pmatrix}
0_n & J_n & J_n & J_{n \times 2n} \\
J_n & 0_n & J_n & J_{n \times 2n} \\
J_n & J_n & 0_n & J_{n \times 2n} \\
J_{2n \times n} & J_{2n \times n} & J_{2n \times n} & 0_{2n}
\end{pmatrix} \otimes J_n = B \otimes J_n,$$

where $J_n$ is the square matrix with all entries one.

**Example 2.3.** Consider now the $NC$-graph of group $T_{4n}$ with the following presentation

$$T_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$ 

The elements of this group are

$$\{1, a, \cdots, a^{2n-1} , b, ba, \cdots, ba^{2n-1}\}.$$ 

One can prove that $Z(T_{4n}) = \langle b^2 \rangle$ and so $|Z(T_{4n})| = 2$. This implies that

$$|V(\Gamma(T_{4n}))| = |T_{4n}| - |Z(T_{4n})| = 4n - 2.$$
It is not difficult to see that, $\Gamma(T_{4n})$ is $(n+1)$-partite graph. On the other hand, $\Gamma$ has $2n - 2$ vertices of degree $2n$ and $2n$ vertices of degree $4n - 4$. This implies that the adjacency matrix of $\Gamma(T_{4n})$ is

$$
\begin{pmatrix}
0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\
\vdots & & & \vdots & & \\
0_2 & \cdots & 0_2 & J_2 & \cdots & J_2 \\
J_2 & \cdots & J_2 & 0_2 & \cdots & J_2 \\
\vdots & & & \vdots & & \\
J_2 & \cdots & J_2 & J_2 & \cdots & 0_2
\end{pmatrix}
= 
\left(\begin{array}{cc}
0_{n-1} & J_{(n-1)\times n} \\
J_{n\times(n-1)} & (J - I)_n
\end{array}\right) \otimes J_2.
$$

We recall that a finite group is called a $p$-group if and only if its order is a power of $p$, where $p$ is a prime integer. In [13], it is proved that there is no regular $NC$-graph of valency $p^n$, where $p$ is an odd prime number and $n$ is a positive integer. In general, we have the following result.

**Theorem 2.4.** [13] Let $G$ be a finite non-abelian group such that $\Gamma(G)$ is $k$-regular. Then $k$ is an even number greater than or equal with 4.

**Theorem 2.5.** [13] Let $G$ be a finite non-abelian group such that $\Gamma(G)$ is $2^s$-regular, where $s \in \mathbb{N} \setminus \{1\}$. Then $G$ is a 2-group.

**Proposition 2.6.** [1] Let $G$ be a finite non-abelian group such that $\Gamma(G)$ is a regular graph. Then $G$ is nilpotent of class at most 3 and $G = P \times A$, where $A$ is an abelian group, $P$ is a $p$-group ($p$ is a prime) and furthermore $\Gamma(P)$ is a regular graph.

**Theorem 2.7.** [13] Let $G$ be a non-abelian group and $p$ be a prime number. If $[G : Z(G)] = p^2$, then $\Gamma(G)$ is a regular graph.

**Theorem 2.8.** [6] Let $G$ be a finite non-abelian group. Then $|\text{Cent}(G)| = 4$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Theorem 2.9.** [6] Let $G$ be a finite non-abelian group and $p$ be a prime number. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $|\text{Cent}(G)| = p + 2$.

**Remark 2.10.** Let $G \cong P \times \mathbb{Z}_q$ where $p, q$ are prime numbers and $P$ be a $p$-group. Hence, we have $G/Z(G) \cong P/Z(P)$. Thus, $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

**Proposition 2.11.** [13] Let $p$ be a prime number and $P$ be a non-abelian $p$-group. Then $\Gamma(P)$ is $k$-regular if and only if $\Gamma(P \times \mathbb{Z}_q)$ is $kq$-regular, where $q$ is a prime number.

In the following by $\overline{K}_n$ we mean the complement of the complete graph $K_n$. 


Corollary 2.12. Let \( p \) be a prime number and \( P \) be a non-abelian \( p \)-group. If \( G = P \times A \), where \( A \) is an abelian group, then the graph \( \Gamma(G) \) is lexicographic product of \( \Gamma(P) \) around \( \overline{K}_{|A|} \) i.e. \( \Gamma(G) \cong \Gamma(P) \circ \overline{K}_{|A|} \).

Theorem 2.13. Let \( G \) be a finite non-abelian group and \( p \) be a prime number. Then \( G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \) if and only if \( \Gamma(G) \) is a regular complete \((p + 1)\)-partite graph.

3. Main Results

Let \( \Gamma \) be a graph with adjacency matrix \( A \), the characteristic polynomial of \( \Gamma \) is defined as \( \chi_{\Gamma}(\lambda) = \det(\lambda I - A) \), where \( I \) is the identity matrix. The roots of this polynomial are called the eigenvalues of \( \Gamma \) and form the spectrum of this graph, see [4, 10, 11, 15, 16]. It is a well-known fact that if \( A \) is a real symmetric matrix, then all eigenvalues of \( A \) are real. The graph \( \Gamma \) is said to be integral if all its eigenvalues are integers, see [3, 4, 11].

Proposition 3.1. A graph has exactly one positive eigenvalue if and only if the non-isolated vertices form a complete multipartite graph.

Lemma 3.2. Let \( M \) be the following block matrix:

\[
M = \begin{pmatrix}
0_{m \times m} & B_{m \times n} \\
B_{n \times m}^T & A_{n \times n}
\end{pmatrix}.
\]

Then

\[
\chi_M(\lambda) = |\lambda I - M| = \lambda^{m-n} |\lambda^2 I_n - \lambda A - B^T B|.
\]

Theorem 3.3. Let \( A \) and \( B \) be square matrices of orders \( m \) and \( n \), respectively. If \( \lambda_1, \ldots, \lambda_m \) are eigenvalues of \( A \) and \( \mu_1, \ldots, \mu_n \) are eigenvalues of \( B \), then for \( 1 \leq i \leq m, 1 \leq j \leq n \), the eigenvalues of \( A \otimes B \) are \( \lambda_i \mu_j \) and the eigenvalues of \( A \otimes I_m + I_n \otimes B \) are \( \lambda_i + \mu_j \).

The aim of this section is to study the spectral properties of \( NC \)-graphs. In [1] it is proved that the diameter of an \( NC \)-graph is two. On the other hand, in [10] it is proved that if \( \Gamma \) is an integral \( k \)-regular graph on \( n \) vertices with diameter \( d \), then

\[
n \leq \frac{k(k-1)^d - 2}{k-2}.
\]

By using these results, in [13] the authors has proposed a necessary condition for \( \Gamma(G) \) to be an integral \( k \)-regular graph. Here, we give a sufficient condition for \( \Gamma(G) \) to be integral.

Theorem 3.4. Let \( G \) be a finite non-abelian group and \( p \) be a prime number. If \( G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \), then \( \Gamma(G) \) is an integral graph.
Proof. By Theorem 2.13, \( \Gamma(G) \) is a regular complete \((p + 1)\)-partite graph and so it is a strongly regular graph with parameters \((k, \lambda, \mu)\). Hence, the eigenvalues of \( \Gamma(G) \) are as follows:

\[
\left\{ \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} \right\}^{m_1}, \left\{ \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} \right\}^{m_2}, [k]^1
\]

If \( n \) is the number of vertices of the graph, then the number of vertices of each part of this graph is \( n = \frac{pn}{p + 1} \). Hence, we have

\[
k = \frac{pm}{p + 1}, \quad \lambda = \frac{(p - 1)n}{p + 1}, \quad \mu = \frac{pn}{p + 1}
\]

Therefore the spectrum of \( \Gamma(G) \) is

\[
\text{Spec}(\Gamma(G)) = \left\{ \left\lfloor \frac{-n}{p + 1} \right\rfloor^{m_1}, [0]^{m_2}, \left\lfloor \frac{pm}{p + 1} \right\rfloor^1 \right\}
\]

On the other hand, \( m_1 + m_2 + 1 = n \) and \( \frac{pm}{p + 1} + m_1 - \frac{n}{p + 1} = 0 \). Hence \( m_1 = p \) and \( m_2 = n - 1 - p \). Since \( p + 1 \) divides \( n \), the eigenvalues of this graph are integral. \( \square \)

Corollary 3.5. Let \( G \) be a finite non-abelian group and \( p \) be a prime number. If \( G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \), then \( \Gamma(G) \) has only one positive eigenvalue. In particular, \( p(p - 1)|Z(G)| \) is the only positive eigenvalue of the regular graph \( \Gamma(G) \).

Proof. According to Theorem 2.13, \( \Gamma(G) \) is a complete \((p + 1)\)-partite graph. Thus, by using Proposition 3.1, \( \Gamma(G) \) has only one positive eigenvalue. By Theorem 3.4 the positive eigenvalue of \( \Gamma(G) \) is

\[
\frac{p(\text{order of } G - \text{order of } Z(G))}{p + 1} = \frac{p(p^2 - 1)|Z(G)|}{p + 1} = p(p - 1)|Z(G)|.
\]

\( \square \)

Theorem 3.6. Let \( p \) be a prime number and \( P \) be a \( p \)-group. If \( G = P \times A \) where \( A \) is an abelian group, then the spectrum of \( \Gamma(G) \) is

\[
\text{Spec}(\Gamma(G)) = \left\{ [0]^{(a-1)|V(\Gamma(P))|}, [a\lambda_1]^{m_1}, \ldots, [a\lambda_s]^{m_s} \right\},
\]

where \( \{[\lambda_1]^{m_1}, \ldots, [\lambda_s]^{m_s}\} \) is the spectrum of \( \Gamma(P) \) and \( |A| = a \).

Proof. By Corollary 2.12, we have \( \Gamma(G) \cong \Gamma(P) \circ \mathcal{K}_{|A|} \). Let \( |A| = a \) and \( B \) be the adjacency matrix of \( \Gamma(P) \). Since the adjacency matrix of \( \mathcal{K}_{|A|} \) is \((0)_{a \times a}\), the adjacency matrix of \( \Gamma(G) \) is \( B \otimes J_a \). Since the characteristic polynomial of \( J_a \) is \( \chi_{J_a}(\lambda) = \lambda^{a-1}(\lambda - a) \), by using Theorem 3.3, the spectrum of \( \Gamma(G) \) is

\[
\text{Spec}(\Gamma(G)) = \left\{ [0]^{(a-1)|V(\Gamma(P))|}, [a\lambda_1]^{m_1}, \ldots, [a\lambda_s]^{m_s} \right\}.
\]
Lemma 3.7. Consider the block matrix

\[ A = \begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times (n-1)} & (J - I)_n \end{pmatrix}. \]

The characteristic polynomial of \( A \) is

\[ \chi_A(\lambda) = \lambda^{n-2}(\lambda + 1)^{n-1}(\lambda^2 + (1 - n)\lambda - n(n - 1)). \]

Proof. By using Lemma 3.2, we have

\[ \chi_A(\lambda) = \lambda^{m-n}|\lambda^2 I_n - \lambda A - BB^T|, \]

where \( m = n - 1 \) and \( B = J_{n\times (n-1)} \). Hence,

\[ \chi_A(\lambda) = \lambda^{-1}|(\lambda^2 + \lambda)I_n - (\lambda + n - 1)J_n| = \lambda^{-1}(\lambda + n - 1)^n \chi_{\lambda^2 + \lambda} (J_n). \]

Since \( \chi_{J_n}(\lambda) = \lambda^{n-1}(\lambda - n) \), the proof is complete.

Theorem 3.8. The spectrum of \( \Gamma(U_{6n}) \) is

\[ \text{Spec}(\Gamma(U_{6n})) = \left\{ -[n]^2, \left[ n \pm n\sqrt{7} \right]^1, [0]^{5n-4} \right\}. \]

Proof. In Example 2.2, it is shown that \( \Gamma(U_{6n}) \) is a 4-partite graph with the adjacency matrix \( B \otimes J_n \). The eigenvalues of \( J_n \) and \( B \) are \( \{ [0]^{n-1}, [n]^1 \} \) and \( \{ [0]^1, [1 + \sqrt{7}]^1, [1 - \sqrt{7}]^1, [-1]^2 \} \), respectively. Now Theorem 3.3 yeilds the proof.

Theorem 3.9. The spectrum of graph \( \Gamma(T_{4n}) \) is as follows:

\[ \text{Spec}(\Gamma(T_{4n})) = \left\{ [-2]^{n-1}, [0]^{3n-3}, \left( n - 1 \pm \sqrt{(5n - 1)(n - 1)} \right)^1 \right\}. \]

Proof. In Example 2.3, it is shown that \( \Gamma(T_{4n}) \) is a \((n + 1)\)-partite graph with the following adjacency matrix

\[ \begin{pmatrix} 0_{n-1} & J_{(n-1)\times n} \\ J_{n\times (n-1)} & (J - I)_n \end{pmatrix} \otimes J_2. \]

The spectrum of the left hand matrix can be computed directly from Lemma 3.3 as follows:

\[ \left\{ [-1]^{n-1}, [0]^{n-2}, \left[ \frac{n - 1}{2} \pm \frac{\sqrt{(5n - 1)(n - 1)}}{2} \right]^1 \right\}. \]

Thus, by using Theorem 3.3, the proof is complete.
Theorem 3.10. The spectrum of NC-graph $D_{2n}$ is as follows:

if $n$ is odd:

$$\text{Spec}(\Gamma(D_{2n})) = \left\{ [-1]^{n-1}, [0]^{n-2}, \left[ \frac{n-1}{2} \pm \sqrt{\frac{(5n-1)(n-1)}{2}} \right]^1 \right\}.$$ 

if $n$ is even:

$$\text{Spec}(\Gamma(D_{2n})) = \left\{ [-2]^{\frac{n}{2}-1}, [0]^{\frac{2n}{2}-3}, \left[ \frac{n}{2} - 1 \pm \sqrt{\frac{(5n}{2}-1)(n/2-1)} \right]^1 \right\}.$$ 

Proof. In finding the spectrum of $\Gamma(D_{2n})$, it is convenient to consider two separately cases:

Case 1. $n$ is odd, the adjacency matrix of $\Gamma$ has the following form:

$$\begin{pmatrix} 0_{n-1} & J_{(n-1) \times n} \\ J_{n \times (n-1)} & (J - I)_n \end{pmatrix}.$$ 

By using Lemma 3.7, the proof is complete.

Case 2. $n = 2m$ is even, in this case $\Gamma(D_{2n}) \cong \Gamma(T_{4m})$ and according to Theorem 3.9, the proof is complete.

Here, we determine the spectrum of NC-graph of group $V_{8n}(n$ is odd) with the following presentation:

$$V_{8n} = \langle a, b : a^{2n} = b^4 = 1, b^{-1}ab^{-1} = bab = a^{-1} \rangle.$$ 

Theorem 3.11. The spectrum of $\Gamma(V_{8n})$ is given by

$$\left\{ [-2]^{2n-1}, [0]^{6n-3}, \left[ 2n - 1 \pm \sqrt{20n^2 - 12n + 1} \right]^1 \right\}.$$ 

Proof. One can prove that $Z(V_{8n}) = \langle b^2 \rangle$ and so $|Z(V_{8n})| = 2$. This implies that

$$|V(\Gamma(V_{8n}))| = |V_{8n}| - |Z(V_{8n})| = 8n - 2.$$ 

Similar to the proof of Theorem 3.10 and Theorem 3.11, we can show that $\Gamma(V_{8n})$ is a $(2n + 1)$-partite graph with the following partitions:

$$V_1 = \{ a, \ldots, a^{2n-1}, ab^2, \ldots, a^{2n-1}b^2 \},$$

$$V_2 = \{ b, b^3 \},$$

$$V_3 = \{ ab, ab^3 \},$$

$$\vdots$$

$$V_{2n+1} = \{ a^{2n-1}b, a^{2n-1}b^3 \}.$$
In other words, the vertices of \( V_1 \) have degree \( 4n \) and the other vertices have degree \( 8n - 4 \). This implies that its adjacency matrix is \( A(\Gamma(V_{8n})) = C \otimes J_2 \), where

\[
C = \begin{pmatrix}
0_{2n-1} & J_{(2n-1) \times 2n} \\
J_{2n \times (2n-1)} & (J - I)_{2n}
\end{pmatrix}.
\]

By using Lemma 3.7, we have

\[
\chi(C) = \lambda^{2n-2}(\lambda + 1)^{2n-1} (\lambda^2 + (1 - 2n)\lambda - 4n^2 + 2n).
\]

By computing the roots of above polynomial, the spectrum of \( C \) can be computed as follows:

\[
\{ [-1]^{2n-1}, [0]^{6n-3}, [2n - 1 \pm \sqrt{20n^2 - 12n + 1}] \}.
\]

Now, apply Theorem 3.3 to complete the proof.

In continuing, we determine the spectrum of \( NC \)-graph of group \( SD_{8n} \) with the following presentation:

\[ SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle. \]

**Theorem 3.12.** The spectrum of \( \Gamma(SD_{8n}) \) is as follows:

- if \( n \) is even:
  \[ \text{Spec}(\Gamma(SD_{8n})) = \{ [-2]^{2n-1}, [0]^{6n-3}, [2n - 1 \pm \sqrt{20n^2 - 12n + 1}] \} \]

- if \( n \) is odd:
  \[ \text{Spec}(\Gamma(SD_{8n})) = \{ [-4]^{n-1}, [0]^{7n-5}, [2(n - 1) \pm 2\sqrt{(5n - 1)(n - 1)}] \} \]

**Proof.** One can prove that if \( n \) is even, then \( Z(SD_{8n}) = \langle a^{2n} \rangle \) and so \( |Z(SD_{8n})| = 2 \) and if \( n \) is odd, then \( Z(SD_{8n}) = \langle a^n \rangle \). Thus, \( |Z(SD_{8n})| = 4 \). This implies that if \( n \) is even, then

\[ |V(\Gamma(SD_{8n}))| = |SD_{8n}| - |Z(SD_{8n})| = 8n - 2, \]

and if \( n \) is odd, then

\[ |V(\Gamma(SD_{8n}))| = |SD_{8n}| - |Z(SD_{8n})| = 8n - 4. \]

We can show that if \( n \) is even then \( \Gamma(SD_{8n}) \) is a \((2n + 1)\)-partite graph with partitions

\[
\begin{align*}
V_1 &= \{ a, a^2, \ldots, a^{2n-1}, a^{2n+1}, \ldots, a^{4n-1} \}, \\
V_2 &= \{ b, a^{2n}b \}, \\
V_3 &= \{ ab, a^{2n+1}b \}, \\
& \vdots \\
V_{2n+1} &= \{ a^{2n-1}b, a^{4n-1}b \}
\end{align*}
\]
and if \( n \) is odd then \( \Gamma(SD_{8n}) \) is a \((n+1)\)-partite graph with partitions
\[
V_1 = \{a, a^2, \ldots, a^{4n-1}\} \setminus \{a^n, a^{2n}, a^{3n}\},
V_2 = \{b, a^{n}b, a^{2n}b, a^{3n}b\},
V_3 = \{ab, a^{n+1}b, a^{2n+1}b, a^{3n+1}b\},
\vdots
V_{n+1} = \{a^{n-1}b, a^{2n-1}b, a^{3n-1}b, a^{4n-1}b\}.
\]

In other words, if \( n \) is even, then the vertices of \( V_1 \) have degree \( 4n \) and the other vertices have degree \( 8n - 4 \). This implies that its adjacency matrix is equal with \( A(\Gamma(V_{8n})) \) and thus \( \Gamma(SD_{8n}) \) and \( \Gamma(V_{8n}) \), where \( n \) is even, are co-spectral. If \( n \) is odd, the vertices of \( V_1 \) have degree \( 4n \) and the other vertices have degree \( 8n - 8 \). This implies that its adjacency matrix is \( A(\Gamma(SD_{8n})) = C \otimes J_4 \), where
\[
C = \begin{pmatrix}
0 & J_{(n-1)\times n} \\
J_{n\times(n-1)} & (J-I)_n
\end{pmatrix}.
\]
The spectrum of \( C \) can be directly computed by Lemma 3.7 as follows:
\[
\left\{[-1]^{n-1}, [0]^{n-2}, \left[\frac{n-1}{2} \pm \sqrt{(5n-1)(n-1)}\right]\right\}.
\]
Thus, by using Theorem 3.3, the proof is complete.

Finally, we determine the spectrum of \( NC \)-graph of Frobenius group \( F_{p,q} \) in which \( p \) is prime and \( q\mid p-1 \). This group is a non-abelian group of order \( pq \) with the following presentation:
\[
F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u \rangle
\]
where \( u \) is an element of order \( q \) in \( \mathbb{Z}_p^* \).

**Theorem 3.13.** Let \( \alpha = (q-1)(p-1) \). The spectrum of \( \Gamma(F_{p,q}) \) is given by
\[
\left\{[-(q-1)]^{p-1}, [0]^{pq-p-2}, \left[\frac{\alpha \pm \sqrt{\alpha^2 - 4pq}}{2}\right]\right\}.
\]

**Proof.** It is not difficult to see that \( Z(F_{p,q}) = 1 \) and therefore \( |Z(F_{p,q})| = 1 \). The elements of this group are
\[
\{1, a, a^2, \ldots, a^{p-1}\} \cup \{a^mb^n; 0 \leq m \leq p-1, 1 \leq n \leq q-1\}.
\]

Now we compute the centralizer of \( a^mb^n \). First notice that
\[
[G : C_G(a^mb^n)] = |(a^mb^n)^G| = |(b^n)^G| = p.
\]
This implies that \( |G|/|G(a^mb^n)| = p \) and so \( |C_G(a^mb^n)| = q \) which yields \( \langle a^mb^n \rangle \subseteq C_G(a^mb^n) \).

On the other hand, \( o(a^mb^n) = q \) and therefore \( \langle a^mb^n \rangle = C_G(a^mb^n) \). So the graph \( \Gamma(F_{p,q}) \) is a multi-partite graph, where one part is of order \( p-1 \) with the elements \( \{1, a, a^2, \ldots, a^{p-1}\} \) and the other parts are of order \( q-1 \). Clearly, the number of parts of order \( q-1 \) is

\[
\frac{pq - (p-1) - 1}{q-1} = \frac{pq - p}{q-1} = p.
\]

This implies that

\[
A(\Gamma(F_{p,q})) = \begin{pmatrix}
0_{p-1} & J_{(p-1) \times (q-1)} & J_{(p-1) \times (q-1)} & \cdots & J_{(p-1) \times (q-1)} \\
J_{(q-1) \times (p-1)} & 0_{(q-1)} & J_{(q-1)} & \cdots & J_{(q-1)} \\
& \vdots & & & \\
J_{(q-1) \times (p-1)} & J_{(q-1)} & \cdots & \cdots & 0_{(q-1)}
\end{pmatrix}.
\]

This yields that

\[
\det (xI - A(\Gamma(F_{p,q}))) = \begin{vmatrix}
0_{p-1} & J_{(p-1) \times (q-1)} & J_{(p-1) \times (q-1)} & \cdots & J_{(p-1) \times (q-1)} \\
J_{(q-1) \times (p-1)} & 0_{(q-1)} & J_{(q-1)} & \cdots & J_{(q-1)} \\
& \vdots & & & \\
J_{(q-1) \times (p-1)} & J_{(q-1)} & \cdots & \cdots & 0_{(q-1)}
\end{vmatrix} = x^{pq-p-2}(x + (q-1))^{p-1}(x - x_1)(x - x_2),
\]

where

\[
x_1 = \alpha + \sqrt{\alpha^2 - 4\alpha} \quad \text{and} \quad x_2 = \alpha - \sqrt{\alpha^2 - 4\alpha}.
\]

This completes the proof. \( \square \)

\[\text{References}\]


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