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D-SPECTRUM AND D-ENERGY OF COMPLEMENTS OF ITERATED LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The $D-$ eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_p\}$ of a graph G are the eigenvalues of its distance matrix D and form its $D-$ spectrum. The $D-$ energy, $E_D(G)$ of G is given by $E_D(G) = \sum_{i=1}^{p} |\mu_i|$. Two non cospectral graphs with respect to D are said to be $D-$ equi energetic if they have the same $D-$ energy. In this paper we show that if G is an $r-$ regular graph on p vertices with $2r \leq p-1$, then the complements of iterated line graphs of G are of diameter 2 and that $E_D(\overline{L^k(G)})$, $k \geq 2$ depends only on p and r. This result leads to the construction of regular $D-$ equi energetic pair of graphs.

1. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and size (= number of edges) q . The distance matrix $D = D(G)$ of G is defined so that its $(i, j)-$ entry is equal to $d_G(v_i, v_j)$, the distance (= length of the shortest path [3]) between the vertices v_i and v_j of G. The diameter of G denoted by $d(G)$, is the maximum distance between any pair of vertices

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of G [3]. The eigenvalues of $D(G)$ are said to be the $D-$ eigenvalues of G and form the $D-$ spectrum of G, denoted by $spec_D(G)$.

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [5]. In what follows we denote the ordinary eigenvalues of the graph G by $\lambda_i,\ i = 1, 2, \ldots, p$, and the respective spectrum by $spec(G)$. Since the adjacency matrix is real symmetric, the eigenvalues are real and can be labelled so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$.

In a similar manner as the distance matrix is symmetric, all its eigenvalues $\mu_i,\ i = 1, 2, \ldots, p$, are real and can be labeled so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$.

Two graphs G and H for which $spec_D(G) = spec_D(H)$ are said to be $D-$ cospectral. Otherwise, they are non$-D-$ cospectral.

The $D-$ energy, $E_D(G)$, of G [12] is defined as

$$E_D\left(G\right) = \sum_{i=1}^{p} |\mu_i|. \tag{1}$$

Two graphs with equal $D-$ energy are said to be $D-$ equienergetic. $D-$ cospectral graphs are evidently D-equienergetic. Therefore, in what follows we focus our attention to $D-$ equienergetic non $-D-$ cospectral graphs.

The concept of $D-$ energy, Eq. (1), was introduced by Indulal et.al [12]. This definition was motivated by the much older [7] and nowadays extensively studied [9, 10, 11, 14, 15, 17, 18] graph energy, defined in a manner fully analogous to Eq. (1), but in terms of the ordinary graph eigenvalues (eigenvalues of the adjacency matrix, see [5]).

The characteristic polynomial of the $D-$ matrix and the corresponding spectra have been considered in [6, 8]. Moore and Moser in [16] showed for the first time that almost all graphs are of diameter 2 and this result was generalized by Bollobas in [2]. Thus a discussion of graphs of small diameter pertains to almost all graphs.

In [12] the distance spectrum of some graphs of diameter 2 and 3 are established. Also a lower bound for the largest eigenvalue of D, and bounds for the $D-$ energy are obtained. Some pairs of $D-$ equienergetic graphs of diameter 2 , on $p \equiv 1\ mod(3)$ and $p \equiv 0\ mod(6)$ vertices are also constructed. In [13], the distance spectra of some graphs obtained from cycles is derived and a pair of $D-$ equienergetic bipartite graphs on $24t,\ t \geq 3$, vertices is also constructed.

In this paper we discuss the distance spectra of the complements of iterated line graphs of regular graphs and construct one more class of $D-$ equienergetic graphs. For some recent results see [1] and the papers cited there in.

A work of this type is reported here for the first time.

Let G be a graph. Then its complement and line graph are respectively denoted by $\overline{G}$ and $L(G)$ [5].

All graphs considered in this paper are simple, connected and we follow [5] for spectral graph theoretic terminology.

The considerations in the subsequent sections are based on the applications of the following.

Lemma 1.1. [5] *Let G be an $r-$ regular graph. Then r is a simple and the greatest eigenvalue of G.*

Theorem 1.2. [5] *Let G be an $r-$regular graph on p vertices. Then $\overline{G}$ is $p-r-1$ regular and $L(G)$ is $2r-2$ regular. If $\{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$ are the adjacency eigenvalues of G, then*

(1) *The adjacency eigenvalues of $\overline{G}$ are $p-r-1$ and $-1-\lambda_i$, $i = 2, 3, \ldots, p$.*
(2) *The adjacency eigenvalues of $L(G)$ are $2r-2$, $\lambda_i + r - 2$, $i = 2, 3, \ldots, p$ and -2 with multiplicity $\dfrac{p(r-2)}{2}$.*

Theorem 1.3. [12] *Let G be an $r-$ regular graph of order p and $d(G) = 2$. If $\{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$ are its adjacency eigenvalues, then its $D-$ eigenvalues are $2p-r-2$ and $-(\lambda_i + 2)$, $i = 2, 3, \ldots, p$.*

Lemma 1.4. [4] *Let G be an $r-$ regular graph on p vertices. Let p_k and r_k denote the order and degree of its k^{th} iterated line graph $L^k(G)$. Then*

$$p_k = \frac{p}{2^k} \prod_{i=0}^{k-1} \left(2^i r - 2^{i+1} + 2\right),$$
$$r_k = 2^k r - 2^{k+1} + 2.$$

2. On graph complements with diameter 2

In this section we prove the following theorem.

Theorem 2.1. *Let G be an $r-$ regular graph on p vertices. If $r \leq \dfrac{p-1}{2}$, then $d(\overline{G}) = 2$.*

Proof: Let u and v be two vertices of G. If u not adjacent to v in G, then $d(u, v) = 1$ in $\overline{G}$. Now let u adjacent to v in G. Then u and v are collectively adjacent to atmost $2r-2$ vertices other than themselves. Hence they are collectively adjacent to atmost $p-3$ vertices. Therefore always there exists a vertex w in G not adjacent to u and v. Thus in $\overline{G}$, w is adjacent to both u and v and hence $d(u, v) = 2$. Hence the theorem follows.

Theorem 2.2. *Let G be an $r-$ regular graph on p vertices with $r \leq \dfrac{p-1}{2}$. Then $d(\overline{L^k(G)}) = 2$ for all $k \geq 1$.*

Proof: Let G be an $r-$ regular graph on p vertices, $p \geq 8$. Let p_k and r_k respectively denote the order and regularity of the k^{th} iterated line graph $L^k(G)$ of G. Then by Lemma

1.4

$$p_k = \frac{p}{2^k}\prod_{i=0}^{k-1}\left(2^i r - 2^{i+1} + 2\right),$$
$$r_k = 2^k r - 2^{k+1} + 2.$$

Now we shall prove that

(2) $$p_k - 1 - 2r_k \geq 0\,,$$

for all $k \geq 1$ and $2r \leq (p-1)$

We have

$$\begin{aligned} p_k &= \frac{p}{2^k}\prod_{i=1}^{k-1}\left(2^i r - 2^{i+1} + 2\right) \\ &= \frac{p}{2^{k-1}}\prod_{i=1}^{k-2}\left(2^i r - 2^{i+1} + 2\right)\frac{1}{2}\left(2^{k-1} r - 2^k + 2\right) \\ &= p_{k-1}\left(2^{k-2} r - 2^{k-1} + 1\right). \\ r_k &= 2^k r - 2^{k+1} + 2. \\ p_k - 1 - 2r_k &= \left(p_{k-1}\left(2^{k-2} r - 2^{k-1} + 1\right) - 1\right) - 2\left(2^k r - 2^{k+1} + 2\right) \\ &= p_{k-1}\left(2^{k-2} r - 2^{k-1} + 1\right) - 1 - 8\left(2^{k-2} r - 2^{k-1} + 1 - \frac{1}{2}\right) \\ &= p_{k-1} t - 1 - 8\left(t - \frac{1}{2}\right) \\ &= \left(p_{k-1} - 8\right) t + 3 \geq 0, \text{ since } \ p_{k-1} \geq p \geq 8 \text{ and } t = 2^{k-2} r - 2^{k-1} + 1 \geq 0. \end{aligned}$$

Thus $r_k \leq \dfrac{p_k - 1}{2}$ and hence by Theorem 2.1, $d(\overline{L^k(G)}) = 2$.

Lemma 2.3. *Let G be an $r-$ regular graph on p vertices. Let $\{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$ are the adjacency eigenvalues of G. If $d(\overline{G}) = 2$, then the $D-$ eigenvalues of $\overline{G}$ are $\{p + r - 1, \lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_p - 1\}$.*

Proof: Lemma follows from Theorems 1.2 and 1.3.

Theorem 2.4. *Let G be an $r-$ regular graph on p vertices with $r \leq \dfrac{p-1}{2}$. Then $E_D(\overline{L^2(G)}) = 3pr(r-2)$.*

Proof: By Theorem 1.2, the adjacency eigenvalues of $L^2(G)$ are

$$\begin{pmatrix} 4r-6 & \lambda_2 + 3r - 6 & \ldots & \lambda_p + 3r - 6 & 2r - 6 & -2 \\ 1 & 1 & \ldots & 1 & \dfrac{p(r-2)}{2} & \dfrac{pr(r-2)}{2} \end{pmatrix}.$$

Now by Theorem 2.2, $d(\overline{L^2(G)}) = 2$. Then by Lemma 2.3, the $D-$ eigenvalues of $\overline{L^2(G)}$ are

$$\begin{pmatrix} \frac{pr(r-1)}{2} + 4r - 7 & \lambda_2 + 3r - 7 & \dots & \lambda_p + 3r - 7 & 2r - 7 & -3 \\ 1 & 1 & \dots & 1 & \frac{p(r-2)}{2} & \frac{pr(r-2)}{2} \end{pmatrix}.$$

Now by the ordering of eigenvalues λ_i as $r = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$, we have for $r \geq 4$, the only negative $D-$ eigenvalue of $\overline{L^2(G)}$ is -3 with multiplicity $\frac{pr(r-2)}{2}$. Thus
$E_D(\overline{L^2(G)}) = 3pr(r-2)$.

Corollary 1. *Let G be an $r-$ regular graph on p vertices with with $r \leq \frac{p-1}{2}$. Let p_k and r_k denote the order and degree of its k^{th} iterated line graph $L^k(G)$. Then*

$$E_D(\overline{L^k(G)}) = 3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} \left(2^i r - 2^{i+1} + 2\right).$$

Proof:

Since $L^k(G) = L^2(L^{k-2}(G))$, from Theorem 2.4 we have

$$E_D(\overline{L^k(G)}) = 3p_{k-2}r_{k-2}(r_{k-2} - 2), \quad k \geq 3. \tag{3}$$

Now by Lemma 1.4

$$p_{k-2} = \frac{p}{2^{k-2}} \prod_{i=0}^{k-3} \left(2^i r - 2^{i+1} + 2\right) r_{k-2} = 2^{k-2}r - 2^{k-1} + 2.$$

Now substituting Equation 4 in Equation 3 we obtain

$$E_D(\overline{L^k(G)}) = 3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} \left(2^i r - 2^{i+1} + 2\right).$$

Theorem 2.5. *Let G_1 and G_2 be two non $D-$cospectral regular graphs of the same order p and of the same degree r. Let $2r \leq p - 1$. Then for any $k \geq 2$, $r \geq 4$, the graphs $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non $D-$cospectral, $D-$ equienergetic graphs of equal order and of equal number of edges with energy* $3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} \left(2^i r - 2^{i+1} + 2\right)$.

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