



## D-SPECTRUM AND D-ENERGY OF COMPLEMENTS OF ITERATED LINE GRAPHS OF REGULAR GRAPHS

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ABSTRACT. The  $D$ - eigenvalues  $\{\mu_1, \mu_2, \dots, \mu_p\}$  of a graph  $G$  are the eigenvalues of its distance matrix  $D$  and form its  $D$ - spectrum. The  $D$ - energy,  $E_D(G)$  of  $G$  is given by  $E_D(G) = \sum_{i=1}^p |\mu_i|$ . Two non cospectral graphs with respect to  $D$  are said to be  $D$ - equi energetic if they have the same  $D$ - energy. In this paper we show that if  $G$  is an  $r$ - regular graph on  $p$  vertices with  $2r \leq p - 1$ , then the complements of iterated line graphs of  $G$  are of diameter 2 and that  $E_D(\overline{L^k(G)})$ ,  $k \geq 2$  depends only on  $p$  and  $r$ . This result leads to the construction of regular  $D$ - equi energetic pair of graphs.

### 1. INTRODUCTION

Let  $G$  be a connected graph with vertex set  $V(G) = \{v_1, v_2, \dots, v_p\}$  and size (= number of edges)  $q$ . The distance matrix  $D = D(G)$  of  $G$  is defined so that its  $(i, j)$ - entry is equal to  $d_G(v_i, v_j)$ , the distance (= length of the shortest path [3]) between the vertices  $v_i$  and  $v_j$  of  $G$ . The diameter of  $G$  denoted by  $d(G)$ , is the maximum distance between any pair of vertices

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of  $G$  [3]. The eigenvalues of  $D(G)$  are said to be the  $D$ - eigenvalues of  $G$  and form the  $D$ - spectrum of  $G$ , denoted by  $spec_D(G)$ .

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [5]. In what follows we denote the ordinary eigenvalues of the graph  $G$  by  $\lambda_i$ ,  $i = 1, 2, \dots, p$ , and the respective spectrum by  $spec(G)$ . Since the adjacency matrix is real symmetric, the eigenvalues are real and can be labelled so that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p$ .

In a similar manner as the distance matrix is symmetric, all its eigenvalues  $\mu_i$ ,  $i = 1, 2, \dots, p$ , are real and can be labeled so that  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ .

Two graphs  $G$  and  $H$  for which  $spec_D(G) = spec_D(H)$  are said to be  $D$ - cospectral. Otherwise, they are non- $D$ - cospectral.

The  $D$ - energy,  $E_D(G)$ , of  $G$  [12] is defined as

$$(1) \quad E_D(G) = \sum_{i=1}^p |\mu_i|.$$

Two graphs with equal  $D$ - energy are said to be  $D$ - equienergetic.  $D$ - cospectral graphs are evidently  $D$ -equienergetic. Therefore, in what follows we focus our attention to  $D$ - equienergetic non - $D$ - cospectral graphs.

The concept of  $D$ - energy, Eq. (1), was introduced by Indulal et.al [12]. This definition was motivated by the much older [7] and nowadays extensively studied [9, 10, 11, 14, 15, 17, 18] graph energy, defined in a manner fully analogous to Eq. (1), but in terms of the ordinary graph eigenvalues (eigenvalues of the adjacency matrix, see [5]).

The characteristic polynomial of the  $D$ - matrix and the corresponding spectra have been considered in [6, 8]. Moore and Moser in [16] showed for the first time that almost all graphs are of diameter 2 and this result was generalized by Bollobas in [2]. Thus a discussion of graphs of small diameter pertains to almost all graphs.

In [12] the distance spectrum of some graphs of diameter 2 and 3 are established. Also a lower bound for the largest eigenvalue of  $D$ , and bounds for the  $D$ - energy are obtained. Some pairs of  $D$ - equienergetic graphs of diameter 2, on  $p \equiv 1 \pmod{3}$  and  $p \equiv 0 \pmod{6}$  vertices are also constructed. In [13], the distance spectra of some graphs obtained from cycles is derived and a pair of  $D$ - equienergetic bipartite graphs on  $24t$ ,  $t \geq 3$ , vertices is also constructed.

In this paper we discuss the distance spectra of the complements of iterated line graphs of regular graphs and construct one more class of  $D$ - equienergetic graphs. For some recent results see [1] and the papers cited there in.

A work of this type is reported here for the first time.

Let  $G$  be a graph. Then its complement and line graph are respectively denoted by  $\overline{G}$  and  $L(G)$  [5].

All graphs considered in this paper are simple, connected and we follow [5] for spectral graph theoretic terminology.

The considerations in the subsequent sections are based on the applications of the following.

**Lemma 1.1.** [5] *Let  $G$  be an  $r$ -regular graph. Then  $r$  is a simple and the greatest eigenvalue of  $G$ .*

**Theorem 1.2.** [5] *Let  $G$  be an  $r$ -regular graph on  $p$  vertices. Then  $\overline{G}$  is  $p - r - 1$  regular and  $L(G)$  is  $2r - 2$  regular. If  $\{r, \lambda_2, \lambda_3, \dots, \lambda_p\}$  are the adjacency eigenvalues of  $G$ , then*

- (1) *The adjacency eigenvalues of  $\overline{G}$  are  $p - r - 1$  and  $-1 - \lambda_i, i = 2, 3, \dots, p$ .*
- (2) *The adjacency eigenvalues of  $L(G)$  are  $2r - 2, \lambda_i + r - 2, i = 2, 3, \dots, p$  and  $-2$  with multiplicity  $\frac{p(r - 2)}{2}$ .*

**Theorem 1.3.** [12] *Let  $G$  be an  $r$ -regular graph of order  $p$  and  $d(G) = 2$ . If  $\{r, \lambda_2, \lambda_3, \dots, \lambda_p\}$  are its adjacency eigenvalues, then its  $D$ -eigenvalues are  $2p - r - 2$  and  $-(\lambda_i + 2), i = 2, 3, \dots, p$ .*

**Lemma 1.4.** [4] *Let  $G$  be an  $r$ -regular graph on  $p$  vertices. Let  $p_k$  and  $r_k$  denote the order and degree of its  $k^{th}$  iterated line graph  $L^k(G)$ . Then*

$$p_k = \frac{p}{2^k} \prod_{i=0}^{k-1} (2^i r - 2^{i+1} + 2),$$

$$r_k = 2^k r - 2^{k+1} + 2.$$

## 2. ON GRAPH COMPLEMENTS WITH DIAMETER 2

In this section we prove the following theorem.

**Theorem 2.1.** *Let  $G$  be an  $r$ -regular graph on  $p$  vertices. If  $r \leq \frac{p-1}{2}$ , then  $d(\overline{G}) = 2$ .*

**Proof:** Let  $u$  and  $v$  be two vertices of  $G$ . If  $u$  not adjacent to  $v$  in  $G$ , then  $d(u, v) = 1$  in  $\overline{G}$ . Now let  $u$  adjacent to  $v$  in  $G$ . Then  $u$  and  $v$  are collectively adjacent to atmost  $2r - 2$  vertices other than themselves. Hence they are collectively adjacent to atmost  $p - 3$  vertices. Therefore always there exists a vertex  $w$  in  $G$  not adjacent to  $u$  and  $v$ . Thus in  $\overline{G}$ ,  $w$  is adjacent to both  $u$  and  $v$  and hence  $d(u, v) = 2$ . Hence the theorem follows.

**Theorem 2.2.** *Let  $G$  be an  $r$ -regular graph on  $p$  vertices with  $r \leq \frac{p-1}{2}$ . Then  $d(\overline{L^k(G)}) = 2$  for all  $k \geq 1$ .*

**Proof:** Let  $G$  be an  $r$ -regular graph on  $p$  vertices,  $p \geq 8$ . Let  $p_k$  and  $r_k$  respectively denote the order and regularity of the  $k^{th}$  iterated line graph  $L^k(G)$  of  $G$ . Then by Lemma

1.4

$$p_k = \frac{p}{2^k} \prod_{i=0}^{k-1} (2^i r - 2^{i+1} + 2),$$

$$r_k = 2^k r - 2^{k+1} + 2.$$

Now we shall prove that

$$(2) \quad p_k - 1 - 2r_k \geq 0,$$

for all  $k \geq 1$  and  $2r \leq (p - 1)$

We have

$$p_k = \frac{p}{2^k} \prod_{i=1}^{k-1} (2^i r - 2^{i+1} + 2)$$

$$= \frac{p}{2^{k-1}} \prod_{i=1}^{k-2} (2^i r - 2^{i+1} + 2) \frac{1}{2} (2^{k-1} r - 2^k + 2)$$

$$= p_{k-1} (2^{k-2} r - 2^{k-1} + 1).$$

$$r_k = 2^k r - 2^{k+1} + 2.$$

$$p_k - 1 - 2r_k = (p_{k-1} (2^{k-2} r - 2^{k-1} + 1) - 1) - 2(2^k r - 2^{k+1} + 2)$$

$$= p_{k-1} (2^{k-2} r - 2^{k-1} + 1) - 1 - 8 \left( 2^{k-2} r - 2^{k-1} + 1 - \frac{1}{2} \right)$$

$$= p_{k-1} t - 1 - 8 \left( t - \frac{1}{2} \right)$$

$$= (p_{k-1} - 8) t + 3 \geq 0, \text{ since } p_{k-1} \geq p \geq 8 \text{ and } t = 2^{k-2} r - 2^{k-1} + 1 \geq 0.$$

Thus  $r_k \leq \frac{p_k - 1}{2}$  and hence by Theorem 2.1,  $d(\overline{L^k(G)}) = 2$ .

**Lemma 2.3.** *Let  $G$  be an  $r$ -regular graph on  $p$  vertices. Let  $\{r, \lambda_2, \lambda_3, \dots, \lambda_p\}$  are the adjacency eigenvalues of  $G$ . If  $d(\overline{G}) = 2$ , then the  $D$ -eigenvalues of  $\overline{G}$  are  $\{p + r - 1, \lambda_2 - 1, \lambda_3 - 1, \dots, \lambda_p - 1\}$ .*

**Proof:** Lemma follows from Theorems 1.2 and 1.3.

**Theorem 2.4.** *Let  $G$  be an  $r$ -regular graph on  $p$  vertices with  $r \leq \frac{p-1}{2}$ . Then  $E_D(\overline{L^2(G)}) = 3pr(r-2)$ .*

**Proof:** By Theorem 1.2, the adjacency eigenvalues of  $L^2(G)$  are

$$\left( \begin{array}{cccccc} 4r - 6 & \lambda_2 + 3r - 6 & \dots & \lambda_p + 3r - 6 & 2r - 6 & -2 \\ 1 & 1 & \dots & 1 & \frac{p(r-2)}{2} & \frac{pr(r-2)}{2} \end{array} \right).$$

Now by Theorem 2.2,  $d(\overline{L^2(G)}) = 2$ . Then by Lemma 2.3, the  $D$ - eigenvalues of  $\overline{L^2(G)}$  are

$$\left( \begin{array}{cccccc} \frac{pr(r-1)}{2} + 4r - 7 & \lambda_2 + 3r - 7 & \dots & \lambda_p + 3r - 7 & 2r - 7 & -3 \\ 1 & 1 & \dots & 1 & \frac{p(r-2)}{2} & \frac{pr(r-2)}{2} \end{array} \right).$$

Now by the ordering of eigenvalues  $\lambda_i$  as  $r = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p$ , we have for  $r \geq 4$ , the only negative  $D$ - eigenvalue of  $\overline{L^2(G)}$  is  $-3$  with multiplicity  $\frac{pr(r-2)}{2}$ . Thus

$$E_D(\overline{L^2(G)}) = 3pr(r-2).$$

**Corollary 1.** *Let  $G$  be an  $r$ - regular graph on  $p$  vertices with  $r \leq \frac{p-1}{2}$ . Let  $p_k$  and  $r_k$  denote the order and degree of its  $k^{th}$  iterated line graph  $L^k(G)$ . Then*

$$E_D(\overline{L^k(G)}) = 3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} (2^i r - 2^{i+1} + 2).$$

**Proof:**

Since  $L^k(G) = L^2(L^{k-2}(G))$ , from Theorem 2.4 we have

$$(3) \quad E_D(\overline{L^k(G)}) = 3p_{k-2}r_{k-2}(r_{k-2} - 2), \quad k \geq 3.$$

Now by Lemma 1.4

$$p_{k-2} = \frac{p}{2^{k-2}} \prod_{i=0}^{k-3} (2^i r - 2^{i+1} + 2) r_{k-2} = 2^{k-2}r - 2^{k-1} + 2.$$

Now substituting Equation 4 in Equation 3 we obtain

$$E_D(\overline{L^k(G)}) = 3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} (2^i r - 2^{i+1} + 2).$$

**Theorem 2.5.** *Let  $G_1$  and  $G_2$  be two non  $D$ -cospectral regular graphs of the same order  $p$  and of the same degree  $r$ . Let  $2r \leq p-1$ . Then for any  $k \geq 2$ ,  $r \geq 4$ , the graphs  $\overline{L^k(G_1)}$  and  $\overline{L^k(G_2)}$  form a pair of non  $D$ -cospectral,  $D$ - equienergetic graphs of equal order and of equal number of edges with energy  $3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} (2^i r - 2^{i+1} + 2)$ .*

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