



## ON TWO-DIMENSIONAL CAYLEY GRAPHS

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ABSTRACT. A subset  $W$  of the vertices of a graph  $G$  is a resolving set for  $G$  when for each pair of distinct vertices  $u, v \in V(G)$  there exists  $w \in W$  such that  $d(u, w) \neq d(v, w)$ . The cardinality of a minimum resolving set for  $G$  is the metric dimension of  $G$ . This concept has applications in many diverse areas including network discovery, robot navigation, image processing, combinatorial search and optimization. The problem of finding metric dimension is NP-complete for general graphs but the metric dimension of trees can be obtained using a polynomial time algorithm. In this paper, we investigate the metric dimension of Cayley graphs on dihedral groups and we characterize a family of them.

### 1. INTRODUCTION

Let  $\Gamma = (V, E)$  be a simple and connected graph with vertex set  $V$  and edge set  $E$ . The distance between two vertices  $x, y \in V$  is the length of a shortest path between them and is denoted by  $d(x, y)$ . If  $d(x, y) = 1$ , then for convenient we write  $x \sim y$ . The neighborhood of  $x$  is  $N(x) = \{y : x \sim y\}$ . A walk consists of an alternating sequence of vertices and edges

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consecutive elements of which are incident, that begins and ends with a vertex. A walk is said to be closed if its endpoints are the same. The length of a walk is the number of its edges. An odd walk is a walk whose length is an odd number. It is well known that a graph is bipartite if and only if it does not contain any odd walk. A matching or independent edge set in a graph is a set of edges without common vertices. In a graph of even order  $n = |V(G)|$ , each matching with  $\frac{n}{2}$  edges is called a perfect matching. For an ordered subset  $W = \{w_1, w_2, \dots, w_k\}$  of vertices and a vertex  $v \in V$ , the  $k$ -vector  $r(v|W) := (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$  is called the *metric representation* of  $v$  with respect to  $W$ . The set  $W$  is called a *resolving set* for  $\Gamma$  if distinct vertices of  $\Gamma$  have distinct representations with respect to  $W$ . Each minimum resolving set is a *basis* and the *metric dimension* of  $\Gamma$ ,  $\dim_M(\Gamma)$ , is the cardinality of a basis for  $\Gamma$ . These concepts were introduced by Slater in 1975 when he was working with U.S. Sonar and Coast Guard Loran stations and he described the usefulness of these concepts, (see [15]). Independently, Harary and Melter discovered these concepts, (see [7]). This concept has applications in many areas including network discovery and verification (see [2]), robot navigation (see [11]), problems of pattern recognition and image processing (see [12]), coin weighing problems (see [14]), strategies for the Mastermind game (see [5]), combinatorial search and optimization (see [14]). Finding families of graphs with constant metric dimension or characterizing  $n$ -vertex graphs with a specified metric dimension are fascinating problems and attracts the attention of many researchers. The problem of finding metric dimension is NP-Complete for general graphs but the metric dimension of trees can be obtained using a polynomial time algorithm. It is not hard to see that for each  $n$ -vertex graph  $\Gamma$  we have  $1 \leq \dim_M(\Gamma) \leq n - 1$ . Chartrand et al. in [6] proved that for  $n \geq 2$ ,  $\dim_M(\Gamma) = n - 1$  if and only if  $\Gamma$  is the complete graph  $K_n$ . The metric dimension of each complete  $t$ -partite graph with  $n$  vertices is  $n - t$ . They also provided a characterization of graphs of order  $n$  with metric dimension  $n - 2$ , (see [6]). Graphs of order  $n$  with metric dimension  $n - 3$  are characterized in [9]. Khuller et al. (see [11]) and Chartrand et al. (see [6]) proved that  $\dim_M(\Gamma) = 1$  if and only if  $\Gamma$  is a path  $P_n$ . Salman *et al.* studied this parameter for the Cayley graphs on cyclic groups, (see [13]). Imran studied the metric dimension of barycentric subdivision of Cayley graphs in [8]. Each cycle graph  $C_n$  is a 2-dimensional graph ( $\dim_M(C_n) = 2$ ). All of 2-trees with metric dimension two are characterized in [3]. Moreover, in [11] and [16] some properties of 2-dimensional graphs are obtained.

**Theorem 1.1.** [11] *Let  $\Gamma$  be a 2-dimensional graph. If  $\{u, v\}$  is a basis for  $\Gamma$ , then*

- (1) *there is a unique shortest path  $P$  between  $u$  and  $v$ ,*
- (2) *the degrees of  $u$  and  $v$  are at most three,*
- (3) *the degree of each internal vertex on  $P$  is at most five.*

The *Möbius Ladder* graph  $M_n$  is a cubic circulant graph with an even number  $n$  of vertices formed from an  $n$ -cycle by connecting opposite pairs of vertices in the cycle. For the metric dimension of *Möbius Ladders* we have the following result.

**Theorem 1.2.** [1] *Let  $n \geq 8$  be an even number. The metric dimension of each Möbius Ladder  $M_n$  is 3 or 4. Specially,  $\dim_M(M_n) = 3$  when  $n \equiv 2 \pmod{8}$ .*

Cáceres *et al.* studied the metric dimension of the Cartesian product of graphs. Recall that the *Cartesian product* of two graphs  $G_1$  and  $G_2$ , denoted by  $G_1 \times G_2$ , is a graph with vertex set  $V(G_1) \times V(G_2) := \{(u, v) : u \in V(G_1), v \in V(G_2)\}$ , in which  $(u, v)$  is adjacent to  $(u', v')$  whenever  $u = u'$  and  $vv' \in E(G_2)$ , or  $v = v'$  and  $uu' \in E(G_1)$ .

**Theorem 1.3.** [4] *Let  $P_m$  be a path on  $m \geq 2$  vertices and  $C_n$  be a cycle on  $n \geq 3$  vertices. Then the metric dimension of each prism  $P_m \times C_n$  is given by*

$$\dim_M(P_m \times C_n) = \begin{cases} 2 & n \text{ odd,} \\ 3 & n \text{ even.} \end{cases}$$

Let  $G$  be a group and let  $S$  be a subset of  $G$  that is closed under taking inverse and does not contain the identity element, say  $e$ . Recall that the *Cayley graph*  $\text{Cay}(G, S)$  is a graph whose vertex set is  $G$  and two vertices  $u$  and  $v$  are adjacent in it when  $uv^{-1} \in S$ . Since  $S$  is inverse-closed ( $S = S^{-1}$ ) and does not contain the identity,  $\text{Cay}(G, S)$  is a simple graph. It is well known that  $\text{Cay}(G, S)$  is a connected graph if and only if  $S$  is a generating set for  $G$ . Since  $\text{Cay}(G, S)$  is  $|S|$ -regular, part (2) of Theorem 1.1 directly implies the following result.

**Corollary 1.** *If  $S$  is a subset of  $D_{2n}$  such that  $e \notin S = S^{-1}$  and  $|S| \geq 4$ , then we have*

$$\dim_M(\text{Cay}(D_{2n}, S)) \geq 3.$$

For more results in this subject or related subjects see [6], [8] and [10]. In this paper, we study the metric dimension of Cayley graphs on dihedral groups and we characterize all of Cayley graphs on dihedral groups whose metric dimension is two.

## 2. Main results

At first, we provide two lemmas on dihedral groups and a sharp lower bound for the metric dimension of 3-regular bipartite graphs which will be frequently used in the sequel.

**Lemma 2.1.** *The subset  $\{a^i b, a^j b\}$  is a generating set for dihedral group  $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = e \rangle$  if and only if  $\gcd(n, i - j) = 1$ .*

*Proof.* It is straightforward to see that the subgroup generated by these elements is given by

$$\langle a^i b, a^j b \rangle = \{a^{(i-j)t}, a^{(i-j)t+i} b, a^{(i-j)t+j} b \mid t \in \mathbb{Z}\}.$$

Now since we have  $a \in \langle a^{i-j} \rangle$  if and only if  $\gcd(n, i-j) = 1$ , the result follows.  $\square$

**Lemma 2.2.** *If  $4 \mid n$  and  $\gcd(i-j, n) = 2$ , then  $\{a^{\frac{n}{2}}, a^i b, a^j b\}$  is not a generating set for  $D_{2n}$ .*

*Proof.* Since  $\langle a^{i-j} \rangle$  and  $\langle a^2 \rangle$  are two cyclic subgroups of order  $\frac{n}{2}$  in the cyclic group  $\langle a \rangle$ , we have  $\langle a^2 \rangle = \langle a^{i-j} \rangle \subseteq \langle \{a^{\frac{n}{2}}, a^i b, a^j b\} \rangle$ . Since  $4 \mid n$  we have  $a^{\frac{n}{2}} \in \langle a^2 \rangle$  and hence,  $\langle \{a^{\frac{n}{2}}, a^i b, a^j b\} \rangle = \langle \{a^i b, a^j b\} \rangle$ . Now the result follows from Lemma 2.1.  $\square$

**Lemma 2.3.** *Let  $\Gamma$  be a 3-regular bipartite graph on  $n$  vertices. Then  $\dim_M(\Gamma) \geq 3$ .*

*Proof.* Since  $\Gamma$  is not a path,  $\dim_M(\Gamma)$  is at least two. Suppose that  $\dim_M(\Gamma) = 2$  and let  $W = \{u, v\}$  be a resolving set for  $\Gamma$ . Assume that  $d(u, v) = d$  and  $N(u) = \{u_1, u_2, u_3\}$ . It is easy to see that  $d(u_i, v) \in \{d-1, d, d+1\}$ , for each  $1 \leq i \leq 3$ . If there exist  $1 \leq i < j \leq 3$  such that  $d(u_i, v) = d(u_j, v)$ , then  $r(u_i|W) = r(u_j|W)$ , which is a contradiction. Hence, without loss of generality, we can assume that

$$d(u_1, v) = d-1, \quad d(u_2, v) = d, \quad d(u_3, v) = d+1.$$

Let  $\sigma_1$  be a (shortest) path between two vertices  $u$  and  $v$  of length  $d$ , and  $\sigma_2$  be a (shortest) path between two vertices  $u_2$  and  $v$ . Two paths  $\sigma_1$  and  $\sigma_2$  using the edge  $uu_2$  produce an old closed walk of length  $2d+1$  in  $\Gamma$  which contradicts the fact that  $\Gamma$  is a bipartite graph. For the sharpness of this bound, consider the hypercube  $Q_3 = K_2 \times K_2 \times K_2$ .  $\square$

In Theorem 2.4 we characterize all of Cayley graphs on dihedral groups whose metric dimension is two. Recall that the center of  $D_{2n}$  is  $\langle a^{\frac{n}{2}} \rangle$  when  $n$  is even, otherwise it is the trivial subgroup  $\{e\}$ .

**Theorem 2.4.** *Let  $S$  be a generating subset of  $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = e \rangle$  such that  $e \notin S = S^{-1}$ . Then we have  $\dim_M(\text{Cay}(D_{2n}, S)) = 2$  if and only if one of the following cases occurs.*

- a)  $n = 2$  and  $S \in \{\{a, b\}, \{a, ab\}, \{b, ab\}\}$ ,
- b)  $n \geq 3$  and  $S = \{a^i b, a^j b\}$  with  $\gcd(i-j, n) = 1$ ,
- c)  $n \geq 3$  is odd and  $S = \{a^i, a^{-i}, a^j b\}$  with  $\gcd(i, n) = 1$  and  $j \in \{1, 2, \dots, n\}$ .

*Proof.* First suppose that  $\dim_M(\text{Cay}(D_{2n}, S)) = 2$ . Since  $D_{2n}$  is not a cyclic group, we have  $|S| \geq 2$ . Also,  $\text{Cay}(D_{2n}, S)$  is  $|S|$ -regular and part (2) of Theorem 1.1 implies that  $|S| \leq 3$ . Thus,  $2 \leq |S| \leq 3$ . If  $|S| = 2$ , then  $\text{Cay}(D_{2n}, S)$  is a connected 2-regular graph (a cycle) and  $\dim_M(\text{Cay}(D_{2n}, S)) = 2$ . Moreover, with the assumption  $S = \{x, y\}$ , since  $S = S^{-1}$  and  $D_{2n}$  is not cyclic, we have  $y \neq x^{-1}$  and  $x^2 = y^2 = e$ . If  $S = \{a^{\frac{n}{2}}, a^j b\}$  for some

$1 \leq j \leq n$ , then the condition  $D_{2n} = \langle S \rangle$  implies that  $n = 2$ ,  $D_{2n} = D_4 = \{e, a, b, ab\}$  and  $S \in \{\{a, b\}, \{a, ab\}, \{b, ab\}\}$  which provides the case (a). Otherwise,  $S = \{a^i b, a^j b\}$  and using Lemma 2.1 we have  $\gcd(i - j, n) = 1$  and this provides the case (b). Now we can assume that  $|S| = 3$ . Since  $S$  is a generating set and  $e \notin S = S^{-1}$ , we consider the following cases.

**Case 1.**  $S = \{a^i, a^{-i}, a^j b\}$ .

Since  $(a^j b)(a^i)^t(a^j b) = a^{-it}$ , the order of  $a^i$  is  $\frac{n}{\gcd(i, n)}$  and  $S$  is a generating set, we have  $\gcd(i, n) = 1$ . Thus  $o(a^i) = n$  and vertices  $a^{ni}, a^{(n-1)i}, \dots, a^{2i}, a^i$  induce an  $n$ -cycle in  $Cay(D_{2n}, S)$ . Since  $a^j \in \langle a^i \rangle$ , there exists  $k \in \{1, 2, \dots, n\}$  such that  $a^j = a^{ki}$ . Therefore  $n$  vertices

$$a^{ki}b, a^{(k+1)i}b, \dots, a^{(k+n-2)i}b, a^{(k+n-1)i}b$$

induce another cycle in  $Cay(D_{2n}, S)$ . Now for each  $1 \leq \ell \leq n$  let  $M_\ell = \{a^{\ell i}, a^{(k+n-\ell)i}b\}$ . Note that  $a^{ni} = e$  and  $M_s \cap M_k = \emptyset$  for each  $s \neq k$ . Since  $a^{\ell i}(a^{(k+n-\ell)i}b)^{-1} = a^{ki}b = a^j b \in S$ , two vertices  $a^{\ell i}$  and  $a^{(k+n-\ell)i}b$  are adjacent in  $Cay(D_{2n}, S)$ . Thus, the edges  $M_1, M_2, \dots, M_n$  provide a perfect matching in  $Cay(D_{2n}, S)$ . Consequently,  $Cay(D_{2n}, S)$  is isomorphic to  $P_2 \times C_n$ . Now Theorem 1.3 implies that  $\dim_M(Cay(D_{2n}, S)) = 2$  if and only if  $n$  is odd. This provides the case (c). In the sequel we will show that other cases for  $S$  are impossible and they will cause some contradictions.

**Case 2.**  $S = \{a^{n/2}, a^i b, a^j b\}$  where  $n$  is an even number.

Let  $x = a^i b$  and  $y = a^j b$ . Since  $a^{n/2}$  is in the center of  $D_{2n}$  and  $o(a^{n/2}) = 2$ , we have  $\langle S \rangle = \langle a^i b, a^j b \rangle \cup a^{n/2} \langle a^i b, a^j b \rangle$ . Hence,  $a \in \langle a^i b, a^j b \rangle$  or  $a \in a^{n/2} \langle a^i b, a^j b \rangle$ . Note that

$$|\langle a^{n/2}, a^i b \rangle| = |\langle a^{n/2}, a^j b \rangle| = 4.$$

Thus,  $a \notin \langle a^{n/2}, a^i b \rangle$  and  $a \notin \langle a^{n/2}, a^j b \rangle$ .

**Subcase 2.1.**  $a \in \langle a^i b, a^j b \rangle$ .

In this case, using Lemma 2.1 we have  $\gcd(i - j, n) = 1$ . Thus,  $o(xy) = o(a^{i-j}) = n$  and  $Cay(D_{2n}, S)$  contains a Hamiltonian cycle (on  $2n$  vertices) as below.

$$e \sim y \sim xy \sim yxy \sim (xy)^2 \sim y(xy)^2 \sim \dots \sim y(xy)^{n-1} \sim (xy)^n = e.$$

For each divisor  $d$  of  $n$  the cyclic group  $\mathbb{Z}_n$  has unique cyclic subgroup of order  $d$ . Since  $\langle a^{i-j} \rangle = \langle a \rangle$  and  $|\langle a^{(i-j)\frac{n}{2}} \rangle| = |\langle a^{\frac{n}{2}} \rangle| = 2$ , we have  $a^{n/2} = (a^{i-j})^{n/2}$ . For each  $1 \leq \ell \leq \frac{n}{2}$  let  $M_\ell = \{(xy)^\ell, (xy)^{\ell+n/2}\}$  and  $T_\ell = \{y(xy)^\ell, y(xy)^{\ell+n/2}\}$ . Note that  $M_s \neq M_k$  and  $T_s \neq T_k$  for each  $s \neq k$ . Also, each  $M_\ell$  is an edge in  $Cay(D_{2n}, S)$  because

$$(xy)^{\ell+n/2}(xy)^{-\ell} = (xy)^{n/2} = (a^{i-j})^{n/2} = a^{n/2} \in S$$

Thus,  $\{M_1, M_2, \dots, M_{\frac{n}{2}}\}$  is a matching in  $X(D_n, S)$ . Similarly,  $\{T_1, T_2, \dots, T_{\frac{n}{2}}\}$  is a matching and hence,  $\{M_1, M_2, \dots, M_{\frac{n}{2}}, T_1, T_2, \dots, T_{\frac{n}{2}}\}$  provides a perfect matching for  $Cay(D_{2n}, S)$ . Therefore, we have a cycle on  $2n$  vertices in which its opposite pairs of vertices are adjacent (see Figure 1 (i)). This implies that  $Cay(D_{2n}, S)$  is a *Möbius Ladder* and by Theorem 1.2,  $dim_M(Cay(D_{2n}, S))$  is 3 or 4, which is a contradiction.

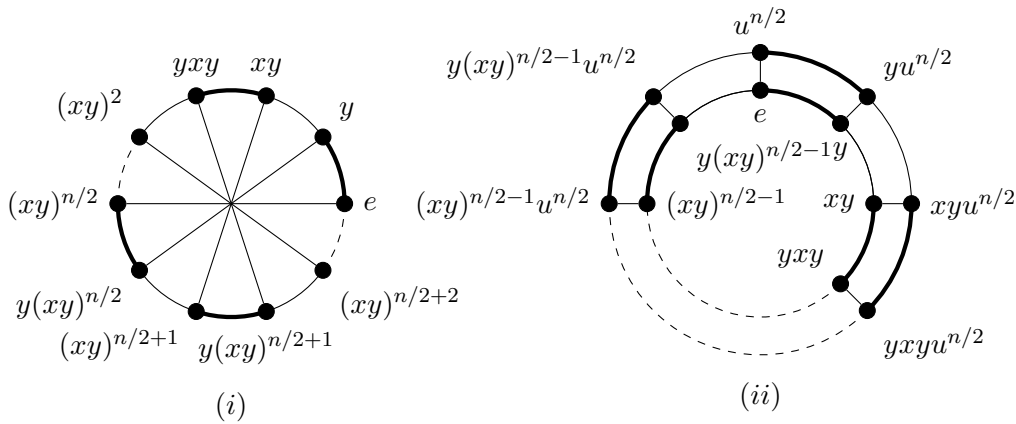


FIGURE 1. (i) A *Möbius Ladder* graph, and (ii) the Cartesian product  $P_2 \times C_n$ .

**Subcase 2.2.**  $a \in a^{n/2}\langle a^i b, a^j b \rangle$ .

In this case, there exists  $k \in \mathbb{Z}$  such that  $a = a^{\frac{n}{2}}(a^{i-j})^k$ . Hence,  $a^{\frac{n}{2}+1} \in \langle a^{i-j} \rangle$  and  $a^2 = (a^{\frac{n}{2}+1})^2 \in \langle a^{i-j} \rangle$ . Thus,  $|\langle a^{i-j} \rangle| \geq |\langle a^2 \rangle| = \frac{n}{2}$ . Hence,  $o(a^{i-j}) = \frac{n}{2}$  or  $o(a^{i-j}) = n$ . The situation  $o(a^{i-j}) = n$  is considered in Subcase 2.1 and we can assume that  $o(xy) = o(a^{i-j}) = n/2$ . Therefore,  $Cay(D_{2n}, S)$  contains two  $n$ -cycles as below.

$$e \sim y \sim xy \sim yxy \sim (xy)^2 \sim y(xy)^2 \sim \dots \sim y(xy)^{n/2-1} \sim (xy)^{n/2} = e,$$

$$a^{n/2} \sim ya^{n/2} \sim (xy)a^{n/2} \sim y(xy)a^{n/2} \sim (xy)^2a^{n/2} \sim \dots \sim (xy)^{n/2}a^{n/2} = a^{n/2}.$$

The fact  $o(xy) = n/2$  implies that  $n$  vertices appeared in each cycle are distinct. Also, using the appearance of  $b$ , it is easy to see that  $(xy)^t \neq y(xy)^s a^{\frac{n}{2}}$  and  $y(xy)^t \neq (xy)^s a^{\frac{n}{2}}$  for each  $t, s \in \mathbb{Z}$ . If there exist  $t, s \in \mathbb{Z}$  such that  $(xy)^t = (xy)^s a^{\frac{n}{2}}$  or  $y(xy)^t = y(xy)^s a^{\frac{n}{2}}$ , then  $a^{\frac{n}{2}} = (a^{i-j})^{(t-s)} \in \langle a^{i-j} \rangle = \langle a^2 \rangle$ . Thus,  $4 \mid n$  which is a contradiction (see Lemma 2.2). Therefore, all of  $2n$  vertices appeared in these cycles are distinct. Since  $a^{\frac{n}{2}} \in S$ , corresponding vertices of two cycles are adjacent (see Figure 1 (ii)). Hence,  $Cay(D_{2n}, S)$  is isomorphic to  $P_2 \times C_n$  and Theorem 1.3 implies that  $dim_M(Cay(D_{2n}, S)) = 3$ . This is a contradiction.

**Case 3.**  $S = \{a^i b, a^j b, a^t b\}$ .

Let  $H = \langle a \rangle$  and hence,  $V(\text{Cay}(D_{2n}, S)) = H \cup Hb$ . If  $a^s, a^t \in H$ , then  $a^s a^{-t} = a^{s-t} \notin S$ . Thus, the subset  $H$  of vertices induces an independent set in  $\text{Cay}(D_{2n}, S)$ . Similarly,  $Hb$  is an independent set. Consequently,  $\text{Cay}(D_{2n}, S)$  is a 3-regular bipartite graph on  $2n$  vertices. Now Lemma 2.3 implies that  $\dim_M(\text{Cay}(D_{2n}, S))$  is at least three, a contradiction.

For the converse suppose that one of the cases (a), (b) or (c) occurs. If case (a) or case (b) occurs, then using Lemma 2.1, the graph  $\text{Cay}(D_{2n}, S)$  is 2-regular and connected. Thus, it is a cycle and its metric dimension is two. If case (c) occurs, then using Case 1 in this proof, we see that  $\text{Cay}(D_{2n}, S)$  is isomorphic to  $P_2 \times C_n$ . Now, Theorem 1.3 implies that  $\dim_M(\text{Cay}(D_{2n}, S)) = 2$  because  $n$  is odd. This completes the proof.  $\square$

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