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ON TWO-DIMENSIONAL CAYLEY GRAPHS

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Abstract. A subset W of the vertices of a graph G is a resolving set for G when for each pair of distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$. The cardinality of a minimum resolving set for G is the metric dimension of G. This concept has applications in many diverse areas including network discovery, robot navigation, image processing, combinatorial search and optimization. The problem of finding metric dimension is NP-complete for general graphs but the metric dimension of trees can be obtained using a polynomial time algorithm. In this paper, we investigate the metric dimension of Cayley graphs on dihedral groups and we characterize a family of them.

1. Introduction

Let $\Gamma = (V, E)$ be a simple and connected graph with vertex set V and edge set E. The distance between two vertices $x, y \in V$ is the length of a shortest path between them and is denoted by $d(x, y)$. If $d(x, y) = 1$, then for convenient we write $x \sim y$. The neighborhood of x is $N(x) = \{y : \ x \sim y\}$. A walk consists of an alternating sequence of vertices and edges

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consecutive elements of which are incident, that begins and ends with a vertex. A walk is said to be closed if its endpoints are the same. The length of a walk is the number of its edges. An odd walk is a walk whose length is an odd number. It is well known that a graph is bipartite if and only if it does not contain any odd walk. A matching or independent edge set in a graph is a set of edges without common vertices. In a graph of even ordr $n = |V(G)|$, each matching with $\frac{n}{2}$ edges is called a perfect matching. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of vertices and a vertex $v \in V$, the k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ is called the *metric representation* of v with respect to W. The set W is called a *resolving set* for Γ if distinct vertices of Γ have distinct representations with respect to W. Each minimum resolving set is a *basis* and the *metric dimension* of Γ, $\dim_M(\Gamma)$, is the cardinality of a basis for Γ. These concepts were introduced by Slater in 1975 when he was working with U.S. Sonar and Coast Guard Loran stations and he described the usefulness of these concepts, (see [15]). Independently, Harary and Melter discovered these concepts, (see [7]). This concept has applications in many areas including network discovery and verification (see [2]), robot navigation (see [11]), problems of pattern recognition and image processing (see [12]), coin weighing problems (see [14]), strategies for the Mastermind game (see [5]), combinatorial search and optimization (see [14]). Finding families of graphs with constant metric dimension or characterizing n-vertex graphs with a specified metric dimension are fascinating problems and attracts the attention of many researchers. The problem of finding metric dimension is NP-Complete for general graphs but the metric dimension of trees can be obtained using a polynomial time algorithm. It is not hard to see that for each n-vertex graph Γ we have $1 \leq \dim_M(\Gamma) \leq n-1$. Chartrand et al. in [6] proved that for $n \geq 2$, $\dim_M(\Gamma) = n-1$ if and only if Γ is the complete graph K_n. The metric dimension of each complete t-partite graph with n vertices is $n-t$. They also provided a characterization of graphs of order n with metric dimension $n-2$, (see [6]). Graphs of order n with metric dimension $n-3$ are characterized in [9]. Khuller et al. (see [11]) and Chartrand et al. (see [6]) proved that $\dim_M(\Gamma) = 1$ if and only if Γ is a path P_n. Salman *et al.* studied this parameter for the Cayley graphs on cyclic groups, (see [13]). Imran studied the metric dimension of barycentric subdivision of Cayley graphs in [8]. Each cycle graph C_n is a 2-dimensional graph ($\dim_M(C_n) = 2$). All of 2-trees with metric dimension two are characterized in [3]. Moreover, in [11] and [16] some properties of 2-dimensional graphs are obtained.

Theorem 1.1. [11] *Let Γ be a 2-dimensional graph. If $\{u, v\}$ is a basis for Γ, then*

(1) *there is a unique shortest path P between u and v,*
(2) *the degrees of u and v are at most three,*
(3) *the degree of each internal vertex on P is at most five.*

The *Möbius Ladder* graph M_n is a cubic circulant graph with an even number n of vertices formed from an n-cycle by connecting opposite pairs of vertices in the cycle. For the metric dimension of *Möbius Ladders* we have the following result.

Theorem 1.2. [1] *Let $n \geq 8$ be an even number. The metric dimension of each Mobius Ladder M_n is 3 or 4. Specially, $dim_M(M_n) = 3$ when $n \equiv 2 \pmod 8$.*

Cáceres *et al*. studied the metric dimension of the Cartesian product of graphs. Recall that the *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with vertex set $V(G_1) \times V(G_2) := \{(u, v) : u \in V(G_1), v \in V(G_2)\}$, in which (u, v) is adjacent to (u', v') whenever $u = u'$ and $vv' \in E(G_2)$, or $v = v'$ and $uu' \in E(G_1)$.

Theorem 1.3. [4] *Let P_m be a path on $m \geq 2$ vertices and C_n be a cycle on $n \geq 3$ vertices. Then the metric dimension of each prism $P_m \times C_n$ is given by*

$$dim_M(P_m \times C_n) = \begin{cases} 2 & n \ odd, \\ 3 & n \ even. \end{cases}$$

Let G be a group and let S be a subset of G that is closed under taking inverse and does not contain the identity element, say e. Recall that the *Cayley graph* $Cay(G, S)$ is a graph whose vertex set is G and two vertices u and v are adjacent in it when $uv^{-1} \in S$. Since S is inverse-closed ($S = S^{-1}$) and does not contain the identity, $Cay(G, S)$ is a simple graph. It is well known that $Cay(G, S)$ is a connected graph if and only if S is a generating set for G. Since $Cay(G, S)$ is $|S|$-regular, part (2) of Theorem 1.1 directly implies the following result.

Corollary 1. If S is a subset of D_{2n} such that $e \notin S = S^{-1}$ and $|S| \geq 4$, then we have

$$\dim_M(Cay(D_{2n}, S)) \geq 3.$$

For more results in this subject or related subjects see [6], [8] and [10]. In this paper, we study the metric dimension of Cayley graphs on dihedral groups and we characterize all of Cayley graphs on dihedral groups whose metric dimension is two.

2. Main results

At first, we provide two lemmas on dihedral groups and a sharp lower bound for the metric dimension of 3-regular bipartite graphs which will be frequently used in the sequel.

Lemma 2.1. *The subset $\{a^i b, a^j b\}$ is a generating set for dihedral group $D_{2n} = \langle a, b |\ a^n = b^2 = (ab)^2 = e\rangle$ if and only if $\gcd(n, i - j) = 1$.*

Proof. It is strightforward to see that the subgroup generated by these elements is given by

$$\langle a^i b, a^j b\rangle = \{a^{(i-j)t},\ a^{(i-j)t+i}b,\ a^{(i-j)t+j}b \mid t \in \mathbb{Z}\}.$$

Now since we have $a \in \langle a^{i-j} \rangle$ if and only if $\gcd(n, i-j) = 1$, the result follows. □

Lemma 2.2. *If* $4 \mid n$ *and* $\gcd(i-j, n) = 2$, *then* $\{a^{\frac{n}{2}}, a^i b, a^j b\}$ *is not a generating set for* D_{2n}.

Proof. Since $\langle a^{i-j} \rangle$ and $\langle a^2 \rangle$ are two cyclic subgroups of order $\frac{n}{2}$ in the cyclic group $\langle a \rangle$, we have $\langle a^2 \rangle = \langle a^{i-j} \rangle \subseteq \langle \{a^{\frac{n}{2}}, a^i b, a^j b\} \rangle$. Since $4 \mid n$ we have $a^{\frac{n}{2}} \in \langle a^2 \rangle$ and hence, $\langle \{a^{\frac{n}{2}}, a^i b, a^j b\} \rangle = \langle \{a^i b, a^j b\} \rangle$. Now the result follows from Lemma 2.1. □

Lemma 2.3. *Let* Γ *be a* 3*-regular bipartite graph on* n *vertices. Then* $dim_M(\Gamma) \geq 3$.

Proof. Since Γ is not a path, $dim_M(\Gamma)$ is at least two. Suppose that $dim_M(\Gamma) = 2$ and let $W = \{u, v\}$ be a resolving set for Γ. Assume that $d(u, v) = d$ and $N(u) = \{u_1, u_2, u_3\}$. It is easy to see that $d(u_i, v) \in \{d-1, d, d+1\}$, for each $1 \leq i \leq 3$. If there exist $1 \leq i < j \leq 3$ such that $d(u_i, v) = d(u_j, v)$, then $r(u_i|W) = r(u_j|W)$, which is a contradiction. Hence, without loss of generality, we can assume that

$$d(u_1, v) = d - 1, \;\; d(u_2, v) = d, \;\; d(u_3, v) = d + 1.$$

Let σ_1 be a (shortest) path between two vertices u and v of lengh d, and σ_2 be a (shortest) path between two vertices u_2 and v. Two paths σ_1 and σ_2 using the edge uu_2 produce an old closed walk of lengh $2d+1$ in Γ which contradicts the fact that Γ is a bipartite graph. For the sharpness of this bound, consider the hypercube $Q_3 = K_2 \times K_2 \times K_2$. □

In Theorem 2.4 we characterize all of Cayley graphs on dihedral groups whose metric dimension is two. Recall that the center of D_{2n} is $\langle a^{\frac{n}{2}} \rangle$ when n is even, otherwise it is the trivial subgroup $\{e\}$.

Theorem 2.4. *Let* S *be a generating subset of* $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = e \rangle$ *such that* $e \notin S = S^{-1}$. *Then we have* $dim_M(Cay(D_{2n}, S)) = 2$ *if and only if one of the following cases occurs.*

a) $n = 2$ *and* $S \in \big\{\{a, b\}, \{a, ab\}, \{b, ab\}\big\}$,
b) $n \geq 3$ *and* $S = \{a^i b, a^j b\}$ *with* $\gcd(i-j, n) = 1$,
c) $n \geq 3$ *is odd and* $S = \{a^i, a^{-i}, a^j b\}$ *with* $gcd(i, n) = 1$ *and* $j \in \{1, 2, ..., n\}$.

Proof. First suppose that $dim_M(Cay(D_{2n}, S)) = 2$. Since D_{2n} is not a cyclic group, we have $|S| \geq 2$. Also, $Cay(D_{2n}, S)$ is $|S|$-regular and part (2) of Theorem 1.1 implies that $|S| \leq 3$. Thus, $2 \leq |S| \leq 3$. If $|S| = 2$, then $Cay(D_{2n}, S)$ is a connected 2-regular graph (a cycle) and $dim_M(Cay(D_{2n}, S)) = 2$. Moreover, with the assumption $S = \{x, y\}$, since $S = S^{-1}$ and D_{2n} is not cyclic, we have $y \neq x^{-1}$ and $x^2 = y^2 = e$. If $S = \{a^{\frac{n}{2}}, a^j b\}$ for some

$1 \leq j \leq n$, then the condition $D_{2n} = \langle S \rangle$ implies that $n = 2$, $D_{2n} = D_4 = \{e, a, b, ab\}$ and $S \in \big\{\{a, b\}, \{a, ab\}, \{b, ab\}\big\}$ which provides the case (a). Otherwise, $S = \{a^i b, a^j b\}$ and using Lemma 2.1 we have $\gcd(i - j, n) = 1$ and this provides the case (b). Now we can assume that $|S| = 3$. Since S is a generating set and $e \notin S = S^{-1}$, we consider the following cases.

Case 1. $S = \{a^i, a^{-i}, a^j b\}$.

Since $(a^j b)(a^i)^t(a^j b) = a^{-it}$, the order of a^i is $\frac{n}{\gcd(i,n)}$ and S is a generating set, we have $gcd(i, n) = 1$. Thus $o(a^i) = n$ and vertices $a^{ni}, a^{(n-1)i}, ..., a^{2i}, a^i$ induce an n-cycle in $Cay(D_{2n}, S)$. Since $a^j \in \langle a^i \rangle$, there exists $k \in \{1, 2, ..., n\}$ such that $a^j = a^{ki}$. Therefore n vertices

$$a^{ki}b, a^{(k+1)i}b, ..., a^{(k+n-2)i}b, a^{(k+n-1)i}b$$

induce another cycle in $Cay(D_{2n}, S)$. Now for each $1 \leq \ell \leq n$ let $M_\ell = \{a^{\ell i}, a^{(k+n-\ell)i}b\}$. Note that $a^{ni} = e$ and $M_s \cap M_k = \emptyset$ for each $s \neq k$. Since $a^{\ell i}(a^{(k+n-\ell)i}b)^{-1} = a^{ki}b = a^j b \in S$, two vertices $a^{\ell i}$ and $a^{(k+n-\ell)i}b$ are adjacent in $Cay(D_{2n}, S)$. Thus, the edges $M_1, M_2, ..., M_n$ provide a perfect matching in $Cay(D_{2n}, S)$. Consequently, $Cay(D_{2n}, S)$ is isomorphic to $P_2 \times C_n$. Now Theorem 1.3 implies that $dim_M(Cay(D_{2n}, S)) = 2$ if and only if n is odd. This provides the case (c). In the sequel we will show that other cases for S are impossimble and they will cause some contradictions.

Case 2. $S = \{a^{n/2}, a^i b, a^j b\}$ where n is an even number.

Let $x = a^i b$ and $y = a^j b$. Since $a^{n/2}$ is in the center of D_{2n} and $o(a^{\frac{n}{2}}) = 2$, we have $\langle S \rangle = \langle a^i b, a^j b \rangle \cup a^{n/2} \langle a^i b, a^j b \rangle$. Hence, $a \in \langle a^i b, a^j b \rangle$ or $a \in a^{n/2} \langle a^i b, a^j b \rangle$. Note that

$$|\langle a^{n/2}, a^i b \rangle| = |\langle a^{n/2}, a^j b \rangle| = 4.$$

Thus, $a \notin \langle a^{n/2}, a^i b \rangle$ and $a \notin \langle a^{n/2}, a^j b \rangle$.

Subcase 2.1. $a \in \langle a^i b, a^j b \rangle$.

In this case, using Lemma 2.1 we have $gcd(i - j, n) = 1$. Thus, $o(xy) = o(a^{i-j}) = n$ and $Cay(D_{2n}, S)$ contains a Hamiltonian cycle (on $2n$ vertices) as below.

$$e \sim y \sim xy \sim yxy \sim (xy)^2 \sim y(xy)^2 \sim \ldots \sim y(xy)^{n-1} \sim (xy)^n = e.$$

For each divisor d of n the cyclic group $\mathbb{Z}_n$ has unique cyclic subgroup of order d. Since $\langle a^{i-j} \rangle = \langle a \rangle$ and $|\langle a^{(i-j)\frac{n}{2}} \rangle| = |\langle a^{\frac{n}{2}} \rangle| = 2$, we have $a^{n/2} = (a^{i-j})^{n/2}$. For each $1 \leq \ell \leq \frac{n}{2}$ let $M_\ell = \big\{(xy)^\ell, (xy)^{\ell+n/2}\big\}$ and $T_\ell = \big\{y(xy)^\ell, y(xy)^{\ell+n/2}\big\}$. Note that $M_s \neq M_k$ and $T_s \neq T_k$ for each $s \neq k$. Also, each M_ℓ is an edge in $Cay(D_{2n}, S)$ because

$$(xy)^{\ell+n/2}(xy)^{-\ell} = (xy)^{n/2} = (a^{i-j})^{n/2} = a^{n/2} \in S$$

Thus, $\{M_1, M_2, ..., M_{\frac{n}{2}}\}$ is a matching in $X(D_n, S)$. Similarly, $\{T_1, T_2, ..., T_{\frac{n}{2}}\}$ is a matching and hence, $\{M_1, M_2, ..., M_{\frac{n}{2}}, T_1, T_2, ..., T_{\frac{n}{2}}\}$ provides a perfect matching for $Cay(D_{2n}, S)$. Therefore, we have a cycle on $2n$ vertices in which its opposite pairs of vertices are adjacent (see Figure 1 (i)). This implies that $Cay(D_{2n}, S)$ is a *Möbius Ladder* and by Theorem 1.2, $dim_M(Cay(D_{2n}, S))$ is 3 or 4, which is a contradiction.

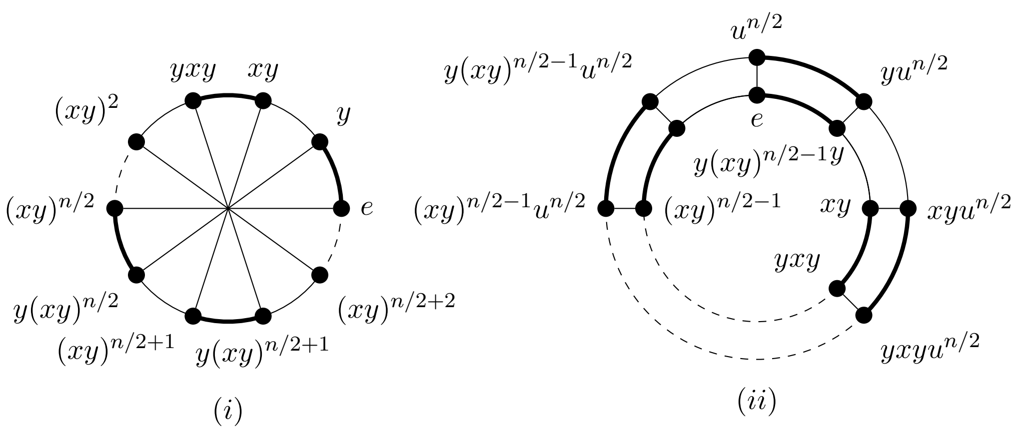

FIGURE 1. (i) A *Möbius Ladder* graph, and (ii) the Cartesian product $P_2 \times C_n$.

Subcase 2.2. $a \in a^{n/2}\langle a^i b, a^j b\rangle$.

In this case, there exists $k \in \mathbb{Z}$ such that $a = a^{\frac{n}{2}}(a^{i-j})^k$. Hence, $a^{\frac{n}{2}+1} \in \langle a^{i-j}\rangle$ and $a^2 = (a^{\frac{n}{2}+1})^2 \in \langle a^{i-j}\rangle$. Thus, $|\langle a^{i-j}\rangle| \geq |\langle a^2\rangle| = \frac{n}{2}$. Hence, $o(a^{i-j}) = \frac{n}{2}$ or $o(a^{i-j}) = n$. The situation $o(a^{i-j}) = n$ is considered in Subcase 2.1 and we can assume that $o(xy) = o(a^{i-j}) = n/2$. Therefore, $Cay(D_{2n}, S)$ contains two n-cycles as below.

$$e \sim y \sim xy \sim yxy \sim (xy)^2 \sim y(xy)^2 \sim \ldots \sim y(xy)^{n/2-1} \sim (xy)^{n/2} = e,$$

$$a^{n/2} \sim ya^{n/2} \sim (xy)a^{n/2} \sim y(xy)a^{n/2} \sim (xy)^2 a^{n/2} \sim \ldots \sim (xy)^{n/2} a^{n/2} = a^{n/2}.$$

The fact $o(xy) = n/2$ implies that n vertices appeared in each cycle are distinct. Also, using the appearence of b, it is easy to see that $(xy)^t \neq y(xy)^s a^{\frac{n}{2}}$ and $y(xy)^t \neq (xy)^s a^{\frac{n}{2}}$ for each $t, s \in \mathbb{Z}$. If there exist $t, s \in \mathbb{Z}$ such that $(xy)^t = (xy)^s a^{\frac{n}{2}}$ or $y(xy)^t = y(xy)^s a^{\frac{n}{2}}$, then $a^{\frac{n}{2}} = (a^{i-j})^{(t-s)} \in \langle a^{i-j}\rangle = \langle a^2\rangle$. Thus, $4 \mid n$ which is a contradiction (see Lemma 2.2). Therefore, all of $2n$ vertices appeared in these cycles are distinct. Since $a^{\frac{n}{2}} \in S$, corresponding vertices of two cycles are adjacent (see Figure 1 (ii)). Hence, $Cay(D_{2n}, S)$ is isomorphic to $P_2 \times C_n$ and Theorem 1.3 implies that $dim_M(Cay(D_{2n}, S)) = 3$. This is a contradiction.

Case 3. $S = \{a^i b, a^j b, a^t b\}$.

Let $H = \langle a \rangle$ and hence, $V(Cay(D_{2n}, S)) = H \cup Hb$. If $a^s, a^t \in H$, then $a^s a^{-t} = a^{s-t} \notin S$. Thus, the subset H of vertices induces an independent set in $Cay(D_{2n}, S)$. Similarly, Hb is an independent set. Consequently, $Cay(D_{2n}, S)$ is a 3-regular bipartite graph on $2n$ vertices. Now Lemma 2.3 implies that $dim_M(Cay(D_{2n}, S))$ is at least three, a contradiction.

For the converse suppose that one of the cases (a), (b) or (c) occurs. If case (a) or case (b) occurs, then using Lemma 2.1, the graph $Cay(D_{2n}, S)$ is 2-regular and connected. Thus, it is is a cycle and its metric dimension is two. If case (c) occurs, then using Case 1 in this proof, we see that $Cay(D_{2n}, S)$ is isomorphic to $P_2 \times C_n$. Now, Theorem 1.3 implies that $dim_M(Cay(D_{2n}, S)) = 2$ because n is odd. This completes the proof. □

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