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THE DISTINGUISHING CHROMATIC NUMBER OF BIPARTITE GRAPHS OF GIRTH AT LEAST SIX

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Abstract. The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a vertex labeling with d labels that is preserved only by a trivial automorphism. The distinguishing chromatic number $\chi_D(G)$ of G is defined similarly, where, in addition, f is assumed to be a proper labeling. We prove that if G is a bipartite graph of girth at least six with the maximum degree $\Delta(G)$, then $\chi_D(G) \leq \Delta(G)+1$. We also obtain an upper bound for $\chi_D(G)$ where G is a graph with at most one cycle. Finally, we state a relationship between the distinguishing chromatic number of a graph and its spanning subgraphs.

1. Introduction

Let $G=(V,E)$ be a simple finite connected graph. We use the standard graph notation. In particular, $\mathrm{Aut}(G)$ denotes the automorphism group of G. The *girth* of a graph G is the length of its shortest cycle, and denoted by $g(G)$. For simple connected graph G, and $v \in V$, the *open neighborhood* of a vertex v is the set $N_G(v)=\{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of a vertex v is the set $N_G[v]=N_G(v)\cup\{v\}$. The *degree* of a vertex v in a graph

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G, denoted by $\deg_G(v)$, is the number of edges of G incident with v. In particular, $\deg_G(v)$ is the number of neighbours of v in G. We denote by $\Delta(G)$ the maximum degree of the vertices of G. A subgraph H of a graph G is said to be *spanning* if $V(G) = V(H)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be *induced* if for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If $S \subseteq V(G)$ is the vertex set of H, then H can be written as $G[S]$ and is said to be induced by S.

An *r-labeling* of the vertices of a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{1, 2, \dots, r\}$. Then the labeling f is called a *proper* r-labeling if any two adjacent vertices have different labels. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum r such that G has a proper r-labeling. A concept of symmetry breaking in a graph was introduced by Albertson and Collins in [1], using the notion of labeling of a graph. An r-labeling is called *distinguishing* if the trivial automorphism is the only automorphism of G that preserves all the vertex labels. The *distinguishing number* of G, denoted $D(G)$, is defined to be the minimum r such that G has a distinguishing r-labeling. Collins and Trenk [2] obtained an analogue of Brooks theorem. It asserts that $D(G) \leq \Delta(G) + 1$ holds for any connected graph G. The equality is achieved if and only if $G = K_{\Delta+1}, K_{\Delta,\Delta}$, or C_5. They also defined the distinguishing chromatic number which incorporates the additional requirement that the labeling be proper. They defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G to be the minimum number of labels needed to properly label the vertices of G so that the only automorphism of G that preserves labels is the identity. Collins and Trenk also proved that $\chi_D(T) \leq \Delta(T) + 1$ for any tree T, where the equality is achieved if and only if T belongs to a special class of trees. In particular, they characterized the connected graphs with maximum possible distinguishing chromatic number, showing that $\chi_D(G) = |V(G)|$ if and only if G is a complete multipartite graph. Further, they showed that for a connected graph G, $\chi_D(G) \leq 2\Delta$ with equality if and only if $G = K_{\Delta,\Delta}$ or $G = C_6$, a cycle of length six. For connected graphs with $\Delta \leq 2$, they also completely determined the distinguishing chromatic number. Laflamme and Seyffarth [5] stated that if G is a connected bipartite graph with maximum degree $\Delta \geq 3$, then $\chi_D(G) \leq 2\Delta - 2$ whenever $G \ncong K_{\Delta-1,\Delta}, K_{\Delta,\Delta}$.

In the next section, we improve this bound when G is a connected bipartite graph with girth at least six. More precisely, we prove that if G is a bipartite graph of girth at least six with the maximum degree $\Delta(G)$, then $\chi_D(G) \leq \Delta(G)+1$. We obtain an upper bound for $\chi_D(G)$ where G is a graph with at most one cycle and a relationship between the distinguishing chromatic number of a graph and its spanning subgraphs in Section 3.

2. Main result

Collins and Trenk [2] proposed the following conjecture:

Conjecture 2.1. [2] *If the girth of a connected graph G is 5 or greater, then $\chi_D(G) \leq \Delta + 1$, where $\Delta \geq 3$.*

Motivated by this conjecture, we show that this conjecture is true for bipartite graphs. For this purpose, we begin with some terminology and background, following [4]. A *rooted tree* (T, z) is a tree T with a distinguished vertex z, the root. The *depth* or *level* of a vertex v is its distance from the root, and the *height* of a rooted tree is the greatest depth in the tree. The *parent* of v is the vertex immediately following v on the unique path from v to the root. Two vertices are *siblings* if they have the same parent.

To prove the main result we state the following observation.

Observation 2.2. Let G be a connected graph with girth at least five.

(i) If x, y, z are the vertices of G such that $x, y \in N(z)$, then $N[x] \cap N[y] = \{z\}$.

(ii) Let x, y, z be the vertices of G such that $x, y \in N^i(z)$, where $N^i(z)$ is the set of all vertices of G at distance i of z. If $x' \in N^{i+1}(z) \cap N(x)$ and $y_1, y_2 \in N^{i+1}(z) \cap N(y)$ such that $y_1 \in N(x')$, then $y_2 \notin N(x')$.

(iii) Let x, y, z, v be the vertices of G such that $x, y, z \in N^i(v)$. If $x' \in N^{i+1}(v) \cap N(x)$, $y_1 \in N^{i+1}(v) \cap N(y)$ and $z_1 \in N^{i+1}(v) \cap N(z)$ such that $y_1 \in N(x')$ and $z_1 \in N(x')$ then $y_1 \notin N(z_1)$.

(iv) Let x, y, z be the vertices of G such that $x, y \in N^i(z)$. If $x' \in N^{i+1}(z) \cap N(x)$ and $y' \in N^{i+1}(z) \cap N(y)$, then $|N^{i+1}(z) \cap N(x') \cap N(y')| \leq 1$.

(v) Let x, y, z be the vertices of G such that $x, y \in N^i(z)$. If $x' \in N^{i+1}(z) \cap N(x)$ and $y' \in N^{i+1}(z) \cap N(y)$, and also x' and y' are adjacent, then $N(x') \cap N(y') = \emptyset$.

Theorem 2.3. *Let G be a connected bipartite graph with girth at least six, $g(G) \geq 6$. If $\Delta \geq 3$, then $\chi_D(G) \leq \Delta + 1$.*

Proof. Let v be a vertex of G with maximum degree $\Delta \geq 3$ with $N(v) = \{v_1, ..., v_\Delta\}$, and (T, v) be a breadth-first search spanning tree rooted at v. By a definition of a breadth-first search spanning tree, the distance between v and any vertex w in T is the same as the distance between v and w in G. Since $g(G) \geq 6$, so the induced subgraphs $G[N_G[v_1] \cup N_G[v_2]]$ and $T[N_G[v_1] \cup N_G[v_2]]$ are the same, by Observation 2.2. For $1 \leq i \leq \Delta$, we suppose that $\deg_G(v_i) = t_i + 1$, and set $N_G(v_i) - \{v\} = \{v_{i1}, \ldots, v_{it_i}\}$. The key point in this proof is that if $N^i(v)$ is the set of all vertices of G at distance i from v, then $N_G(x) \cap N^i(v) = \emptyset$ for every $x \in N^i(v)$ and any $1 \leq i \leq \Delta$, because G does not have an odd cycle. We state our labeling by the following steps:

Step 1) We label the vertex v with 0, and retire the label 0. Thus the vertex v is fixed under each automorphism of G preserving the labeling, and so $N^i(v)$ is mapped to itself, setwise, for every $1 \leq i \leq \Delta$, under each automorphism of G preserving the labeling. Now we label the vertices $v_1, \dots, v_\Delta$ with labels $1, \dots, \Delta$, respectively. Then the vertices $v_1, \dots, v_\Delta$ are fixed under each automorphism of G preserving the labeling.

Step 2) In this step, we label the vertices $N_G(v_i) - \{v\}$ for every $1 \leq i \leq \Delta$, so that each vertex is labeled different from its siblings and its parent in T. For this purpose, we relate the set $A_i = \{1, 2, \dots, \Delta\} - \{i\}$ of labels to the elements of $N_G(v_i) - \{v\}$ for every $1 \leq i \leq \Delta$. Since $g(G) \geq 6$, so for every vertex $x \in N^2(v)$, we have $N(x) \cap N^2(v) = \emptyset$, by Observation 2.2. Hence we can label the vertices of $N_G(v_i) - \{v\}$ with labels in A_i, for every $1 \leq i \leq \Delta$, so that each vertex is labeled different from its siblings in T. Now, since none of vertices in $N^2(v)$ are adjacent, so it can be seen that we have a proper distinguishing labeling of the induced subgraph $G[\{v\} \cup N_G(v) \cup N^2(v)]$ with at most $\Delta + 1$ labels $\{0, 1, 2, \dots, \Delta\}$.

Before we start to label the vertices in $N^3(v)$ we state the following note:

N1. For every $1 \leq i \leq \Delta$, each permutation of labels of vertices in $N_G(v_i) - \{v\}$, or even each new labeling of these vertices with labels in A_i, so that each vertex in $N_G(v_i) - \{v\}$ is labeled different from its siblings in T, makes a new proper distinguishing labeling of the induced subgraph $G[\{v\} \cup N_G(v) \cup N^2(v)]$ with at most $\Delta + 1$ labels.

Step 3) Here we label the vertices in $N^3(v)$. For every $1 \leq i \leq \Delta$ and $1 \leq j \leq t_i$ we define:

$$M_{ij} := N_G(v_{ij}) \cap N^3(v) = \{w_{(ij)k} \mid 1 \leq k \leq m_{ij}\}.$$

Hence $|M_{ij}| = m_{ij}$. We want to label the vertices in M_{ij} distinguishingly with labels $\{1, 2, \dots, \Delta\}$ such that each vertex is labeled different from its neighbours. For this purpose, we suppose that for every $1 \leq i \leq \Delta$, and $1 \leq j \leq t_i$ and $1 \leq k \leq m_{ij}$

$$M_{(ij)k} := N_G(w_{(ij)k}) \cap N^2(v),$$

and $C_{(ij)k}$ is the set of labels are used for elements $M_{(ij)k}$ in Step 2. We relate to vertex $w_{(ij)k}$ the set of labels $B(w_{(ij)k}) := \{1, 2, \dots, \Delta\} \setminus C_{(ij)k}$. If $B(w_{(ij)k}) = \emptyset$, then by N1, we can relabel the vertices of $N^2(v)$ such that $B(w_{(ij)k}) \neq \emptyset$. Hence without loss of generality we can assume that $B(w_{(ij)k}) \neq \emptyset$, for every $1 \leq i \leq \Delta$, $1 \leq j \leq t_i$, and $1 \leq k \leq m_{ij}$. Thus the corresponding label set related to the set M_{ij} is the following set:

$$B_{ij} = \bigcup_{k=1}^{m_{ij}} B(w_{(ij)k}).$$

Before we continue our labeling, we state the two following notes:

N2. If $|M_{(ij)k}| \geq 2$ for some $1 \leq i \leq \Delta$, $1 \leq j \leq t_i$, and $1 \leq k \leq m_{ij}$, then since $g(G) \geq 6$, so the parents of elements in $M_{(ij)k}$ are different in T. On the other hand, since $g(G) \geq 6$, so $|N_G(w_{(ij)k}) \cap N_G(w') \cap N^2(v)| \leq 1$ for all

$w' \in N^3(v) \setminus \{w_{(ij)k}\}$. Hence, the vertex $w_{(ij)k}$ is fixed under each automorphism f of G preserving the labeling, because all vertices in $N^2(v)$ are fixed under f. Therefore, to have a proper distinguishing labeling, it is sufficient to label the vertices $w_{(ij)k}$ different from the label of its neighbours, i.e., with an arbitrary labels in $B(w_{(ij)k})$, because G has no odd cycles and hence $N_G(x) \cap N^3(v) = \emptyset$ for every $x \in N^3(v)$.

N3. Let $M'_{ij} := \{w_{(ij)k_1}, \dots, w_{(ij)k_q}\}$ be the set of all elements of M_{ij} for which $|M_{(ij)k}| = 1$ where $k \in \{k_1, \dots, k_q\} \subseteq \{1, 2, \dots, m_{ij}\}$ for some $1 \le i \le \Delta$, $1 \le j \le t_i$. Then to have a proper distinguishing labeling it is sufficient to label the elements in M'_{ij} different from each other using corresponding label set $B(w_{(ij)k})$ for $k \in \{k_1, \dots, k_q\}$.

Since $N_G(x) \cap N^3(v) = \emptyset$ for every $x \in N^3(v)$, so by N2 and N3, we can label the vertices of each M_{ij} with corresponding labels in B_{ij}, so that the vertices in M'_{ij} have been labeled different from each other, and also the vertices in $M_{ij} \setminus M'_{ij}$, say x, have been labeled with an arbitrary labels in $B(x)$. For instance, if $z \in M_{ij}$ and $|N_G(z) \cap N^2(v)| \le 1$, then we assign the vertex z the smallest label in $B(z)$ not yet assigned to the already-labeled vertices in M'_{ij}. If $|N_G(z) \cap N^2(v)| \ge 2$, then we assign the vertex z an arbitrary label in $B(z)$. Therefore, we have a proper distinguishing labeling of the induced subgraph $G[\{v\} \cup N_G(v) \cup N^2(v) \cup N^3(v)]$, by N2 and N3.

Before we continue, it must be noted that for every $1 \le i \le \Delta$, $1 \le j \le t_i$, each permutation of labels in M'_{ij}, or even each new labeling of these vertices with labels in B_{ij}, so that each vertex in M'_{ij} is labeled different from each other, makes a new proper distinguishing labeling of the induced subgraph $G[\{v\} \cup N_G(v) \cup N^2(v) \cup N^3(v)]$. This argument is satisfied for elements in $M_{ij} \setminus M'_{ij}$, too.

For labeling of vertices in $N^i(v)$, $i \ge 4$, we do the similar steps as for $N^3(v)$ and finally, after the finite number of steps, we obtain a proper distinguishing labeling of G with $\Delta + 1$ labels. □

3. Upper bound of $\chi_D(G)$ for trees and unicyclic graphs

In this section, we show that if the number of cycles in a connected graph is at most one, then the distinguishing chromatic number is at most $\Delta + 1$. We need the following lemma:

Lemma 3.1. [2] *A labeling of a rooted tree (T, z) in which each vertex is labeled differently from its siblings and from its parent is a proper distinguishing labeling.*

Theorem 3.2. *Let G be a connected graph of order n and size m. If $m \le n$ and $\Delta(G) \ge 3$, then $\chi_D(G) \le \Delta + 1$.*

Proof. If $m = n-1$, then G is a tree, and we have the result. Then, we suppose that C is the unique cycle in G with alternative vertices $x_1, \ldots, x_t$. It can be seen that $G - E(C)$ is a forest. In fact, $G - E(C)$ is the union of trees T_{x_i}, $1 \leq i \leq t$, where T_{x_i} has only one common vertex x_i with the cycle C. Since $\chi_D(C_n) \leq 4$, so we can label the vertices of cycle C with labels 0,1,2, and 3 in a proper distinguishing way. It is clear that the cycle C is mapped to itself under each automorphism of G. With respect to the labeling of vertices of C, we conclude that the vertices of C, i.e., $x_1, \ldots, x_t$, are fixed under each automorphism of G preserving that labeling. To label the vertices of trees T_{x_i}, $1 \leq i \leq t$, we note that the vertices $x_1, \ldots, x_t$ are fixed under each automorphism of G preserving that labeling. For every T_{x_i}, we label the adjacent vertices to x_i in T_{x_i} with $\deg_G(x_i) - 2$ distinct labels, except the label of x_i. If these vertices is denoted by $x_{i1}, \ldots, x_{i(\deg_G(x_i)-2)}$, then since the vertices x_{ij}, $1 \leq j \leq \deg_G(x_i) - 2$, has at most $\Delta - 1$ children, so we can label the children of each of x_{ij}, $1 \leq j \leq \deg_G(x_i) - 2$, with distinct labels from the set $\{1, 2, \ldots, \Delta\}$, except the label of vertex x_{ij}. Continuing this process, we can label all vertices of T_{x_i} for $1 \leq i \leq t$. In fact we presented a labeling of the rooted tree (T_{x_i}, x_i) for any $1 \leq i \leq t$, in which each vertex is labeled different from its siblings and from its parent. Hence, we have a proper distinguishing labeling of T_{x_i}, with Δ labels, by Lemma 3.1. Now, with respect to the labeling of cycle C, we conclude that $\chi_D(G) \leq \Delta + 1$. □

4. Conclusion

In 2006, Collins and Trenk proposed a conjecture which state that: If the girth of a connected graph G is 5 or greater, then $\chi_D(G) \leq \Delta+1$, where $\Delta \geq 3$. We proved that if G is a connected bipartite graph with $g(G) \geq 6$ and $\Delta \geq 3$, then $\chi_D(G) \leq \Delta+1$. This shows that the conjecture is true for bipartite graphs. After receiving the referee's report for this paper, we aware of recent work on this conjecture which is proposed by Daniel W. Cranston [3]. Cranston proved that this conjecture is true for all connected graphs with girth at least five. Of course our approach for proving this conjecture for bipartite graphs is different with the approach of Cranston.

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